Accurate Computational Approach on Piecewise Mesh for Singularly Perturbed Differential Equations with Mixed Shifts



A Research Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics (Numerical Analysis)

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Declaration

I undersigned declare that, this research entitled **"Accurate Computational Approach on Piecewise Mesh for Singularly Perturbed Differential Equations with Mixed Shifts"** is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Abstract

This thesis presents an accurate computational approach utilizing piecewise mesh for solving singularly perturbed differential equations with mixed shifts. The proposed method begins by discretizing the solution domain using a defined piecewise mesh size. The singularly perturbed differential problems are first transformed into asymptotically equivalent singularly perturbed boundary value problems. Subsequently, the finite difference approximation technique is employed to convert the derivatives in the differential equations into a linear system of algebraic equations, represented as a three-term recurrence relation that can be effectively solved using the Thomas algorithm. The stability and consistency of the method are rigorously investigated, ensuring the convergence of the proposed approach. Additionally, the influence of delay and advance parameters on the solution profile is explored. To validate the method's applicability, various examples are solved for different perturbation parameters and mesh sizes. The numerical results obtained are compared with existing findings in the literature, demonstrating that the proposed method significantly enhances accuracy. Furthermore, graphical representations illustrate the solution's behavior concerning varying shifts and boundary layer dynamics.

Chapter One
Introduction

1.1. Background of the Study

One of the important subjects in applied mathematics is the theory of singular perturbation problem. Perturbation theory is a subject, which studies the effect of small positive parameter in the mathematical model problems [1]. Mathematically modeled problems refer to a set of consistent equations intended to describe the particular features or behavior of a physical system that we seek to understand. Physical phenomena modeled by differential equations include the prescribed defined conditions [2]. Any differential equation in which the highest order derivative is multiplied by a small positive parameter is called singularly perturbed differential equation. The quantities of interest in many areas of applied mathematics are often to be found as the solution of singularly perturbed differential equations together with the given conditions called singularly perturbed problems. These problems may be contains a delay parameter (negative shifts) and/or advance parameter (positive shift) is known as singularly perturbed differential-difference problems. These types of problems frequently arise in modeling biosciences, vibrational models in control theory, physiological processes, diseases, economics, engineering, and soon [3 - 8].Hence, due to the importance of these problems in real life situations, the need to develop numerical methods for approximation its solution is valuable.

In numerical analysis, finite difference methods are a class of numerical techniques for solving differential equations by approximating derivatives with finite difference approximations. It means that finite difference method obtained by converting differential equation into system equations that can be solved by appropriate iterative techniques. However, for singularly perturbed problems, the existence of small positive parameter in the differential difference equation that yields the problem exhibits boundary or interior layer. Layer is the region at which the solution of singularly perturbed problems changes very quickly in certain small regions, [6].

Hence, it is well known that when the parameter of perturbation is small enough, predictable numerical methods seems difficult to solve the singularly perturbed problems. Thus, it should be important to develop appropriate numerical methods for such problems, whose convergence does

not affected by the perturbation parameter, [6, 7]. Also, as one can see from these books, numerical solutions for any singularly perturbed problems depend on perturbation parameter that causes the solution varies rapidly in some parts and slowly in some other parts of the domain. Due to this, if one body tries to solve these types of problems using standard numerical methods, then less accurate solutions are obtained. Thus, to acquire good accurate numerical results, it is necessary to formulate an alternative numerical method that can handle the considered problems.

There are several numerical methods presented by various authors for solving the singularly perturbed differential equations with mixed shifts. Few and most recent of them are: Galerkin method with exponential fitting [9], Novel approach for Numerical approximation [25], Domain decomposition method [10], fitted numerical scheme [11], and Non-polynomial spline method [12], for solving singularly perturbed differential equations with mixed shifts. Hence, different modified finite difference methods have been adopted for solving the singularly perturbed differential equations with mixed shifts. However, the obtained accuracy of numerical solution takes attention to be work more. Hence, it is necessary to present accurate computational approach on piecewise mesh of the solution domain, which produces more accurate numerical solution for the considered problem.

1.2 Statement of the Problem

So far, different finite difference methods have been adopted for solving the singularly perturbed differential equations with mixed shifts. Nevertheless, the obtained accuracy of numerical solution needs attention to be work more. Thus, one of the drawbacks to most of the presented methods that produce the solution needs attention to work more. Therefore, the main objective of our work is to provide more accurate computational approach on piecewise mesh of the solution domain for solving singularly perturbed differential equations with mixed shifts.

Owing to this, the study intended to answer the following basic research questions:

1. How to formulate accurate computational approach on piecewise mesh for singularly perturbed deferential equation with mixed shifts?
2. How to establish the convergence analysis of the proposed method?
3. To what extent the accuracy and rate of convergence will be yields by the proposed method, are more advantageous than the existing methods?

1.3 Objectives of the Study

1.3.1 General objective

The purpose of this study is to introduce an accurate computational approach on piecewise mesh for solving singularly perturbed differential equations with mixed shifts.

1.3.2. Specific objectives

The specific objectives of the study are:

1. To formulate accurate computational approach on piecewise mesh for solving singularly perturbed differential equations with mixed shifts.
2. To establish the convergence analysis of the proposed method.
3. To assess the extent to which the proposed method improves accuracy and convergence rate compared to existing method.

1.4 Delimitation of the Study

This study delimited to an accurate computational approach on piecewise mesh for solving singularly perturbed differential equations with mixed shifts of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-\delta)+c(x)y(x)+d(x)y(x+\eta)=g(x),\quad x\in(0,1), \qquad (1.1)$$

subject to the interval conditions:

$$\begin{aligned} y(x)&=\phi(x), \quad x\in[-\delta,\,0],\\ y(x)&=\varphi(x), \quad x\in[1,\,1+\eta], \end{aligned} \qquad (1.2)$$

where ε $(0<\varepsilon<1)$ is perturbation parameter, and $\delta,\,\eta$ are left and right shift parameters respectively that satisfying $0<\delta,\,\eta<\varepsilon$. The functions $a(x),\ b(x),\ c(x),\ d(x),\ g(x)$ are assumed sufficiently smooth and bounded for guaranteeing the existence of unique continuous solutions.

1.5 Significance of the Study

The outcomes of this study:

- Produce a more accurate uniformly convergent numerical solution for the considered problem.
- Provide a reference material for scholars to develop higher order numerical methods for solving robin type singularly perturbed reaction diffusion problems.
- Help the graduate students to acquire research skills and scientific procedures.

Chapter Two

Review of Related Literature

2.1 Singular Perturbation Problems and Numerical Methods

Due to the advancement in the field of computational mathematics, numerical methods are most widely utilized to solve the equation arising in the field of applied medical science, engineering and technology. Among the different classification of numerical methods: like: finite difference method, finite element method, Spline approximation method and soon; the finite difference method seems to be the simplest approach for the numerical solution of linear differential equation [14].

Finite difference methods are one of the most widely used numerical method to solve differential equations and their applications in science and technology. In finite difference methods, derivatives appearing in the differential equations are replaced by finite difference approximations obtained by Taylor series expansions at the grid points. This gives a large algebraic system of leaner equations to be solved by Thomas algorism or other methods in place of the differential equations to give the solution value at the grid points and hence the solution is obtained at grid points. Finite difference methods are always convenient choice for solving boundary-valued problems because of their simplicity. Some of the finite difference methods include forward, backward and central finite difference methods.

The problems in which the highest order derivative term of a differential equation is multiplied by a small positive parameter are known to be singularly perturbed problems and the parameter is known as the perturbation parameter. Depending on the solution behavior of the problem in the limiting case when perturbation parameter goes to zero, such type of problems are classified into two classes, namely, (i) regularly perturbed and (ii) singularly perturbed. If the solution of the original problem tends to the solution of the reduced problem (i.e., the problem which is obtained by putting $\varepsilon = 0$ in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed otherwise, it is known as singularly perturbed [15].

In mathematics, more precisely in perturbation theory, a singular perturbation problem is a problem containing a small parameter that cannot be approximated by setting the parameter

value to zero. Singular perturbation problem was first introduced by Prandtl [16] during his report on fluid motion with small friction in a seven page report presented at the third international congress of mathematics in Heidelberg in 1904 in which he demonstrated that fluid flow past over a body can be divided in to two regions, a boundary layer and outer region. However the term "singular perturbation "was first used by Friedrichs, K. and Wasow, W.[17] in a paper presented at a seminar on non- linear vibrations at New York university. The solutions of singular perturbation problems typically contain layers.

Because of the presence of small positive parameter (ε) in the differential equations (DEs), the problem exhibits boundary or interior layer. Up on setting $\varepsilon = 0$, if the order of singularly perturbed differential equations (SPDEs) is reduced by one, then the problem is called convection diffusion type, and if the order is reduced by two, it is called reaction diffusion type. Hence, singularly perturbed differential difference equations (SPDDEs) are types of convection diffusion problem [4]. The convection diffusion type has its different types depending on the kind of layers (boundary and/or interior layers) [5]. Hence, the singularly perturbed convection diffusion boundary valued problems (SPCDBVPs) are divided into problems exhibiting right or left boundary layers, interior layer or boundary and interior layers [6]. Since the solution of this problem exhibits one or two layers, the existing numerical methods give good results only when the mesh size (h) is smaller than the perturbation parameter (ε) (i.e. $h< \varepsilon$) ,but it is an expensive and time consuming process. If we take $h > \varepsilon$, the existing classical numerical methods produce oscillatory solution and pollute the solution in the entire interval, because of boundary layer behavior [4].

The layer resolving numerical methods for SPDEs are usually classified in to the fitted operator and fitted mesh methods [6]. That means the solution of SPPs posses' boundary and/or interior layer. As a result, many of classical methods have severe restrictions on the step-size to preserve the stability properties when the perturbation parameter(s) is very small. To avoid these restrictions, the fitted operator and the fitted mesh methods are designed that gives small truncation errors inside these layers. However, both approaches have their own merits and demerits. Fitted operator methods (FOMs) work well on a uniform mesh but are difficult to extend for higher dimensional problems except the fitted operator methods of non- standard type. Fitted mesh methods (FMMs) are easy to extend for higher dimensional and non-linear problems

but require the prior knowledge of the location and width of the layer [8]. FMMs use non-uniform and non- linear meshes, which will be fine in layer regions and coarse outside the layer regions. The well-known layer resolving fitted meshes are Bakhvalove meshes, which are obtained from some non-linear mesh generating functions, and Shishkin meshes, which are piece wise – uniform meshes[18].

A well-known fact is that the solution of singularly perturbed problems display sharp boundary or interior layers when ε is very small, *i.e.*, the solution varies rapidly in some parts and varies slowly in some other parts. Therefore, the treatment of singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions [19, 20]. Thus, more accurate computational techniques are required to solve singularly perturbed problems.

2.2 Recent Works on Singularly Perturbed Differential equations with Mixed Shifts

Recently, several authors have been trying to develop different numerical methods for solving singularly perturbed differential equations with mixed shifts. Some of the works are:

The authors in [10] developed a domain decomposition method for solving singularly perturbed differential difference equations with mixed shifts. Solutions of which exhibit boundary layer behavior at left end of the interval. A terminal boundary point is introduced into the domain, to decompose it into inner and outer regions. The original problem is reduced to asymptotically equivalent singularly perturbed problems and then transformed into linear system of algebraic equation in the form of three - term recurrence relations, which can be solved by using Thomas algorithm. Using domain decomposition technique with the terminal point, the singularly perturbed problem is treated as inner and outer region problems separately with uniform mesh. The outer region and the modified inner region problems are solved by mixed finite difference method. To validate the applicability of the method model examples have been solved for different values of perturbation parameter and mesh size. The method is repeated for various choices of the terminal point. The obtained numerical results have been compared with some of the previous findings reported in the literature and it is observed that the proposed method is more accurate.

The work referenced in [9] presented exponentially fitted Galerkin Method to solve singularly perturbed differential - difference equations with delay and advance shifts using fitting factor. In order to tackle the terms containing small shifts Taylor approximation is used. The given differential equation was reduced to its convenient form and then transformed into leaner system of algebraic equation in the form of three term recurrence relations, which can be solved by using Thomas algorithm. To validate the applicability of the method model examples have been solved for different values of perturbation parameter and mesh size. The obtained numerical results have been compared with some of the previous findings reported in the literature and it is observed that the proposed method is more accurate.

The authors in [21] presented a finite difference method for solving singularly perturbed differential- difference equations with small shifts of mixed type. They construct a special type of mesh in order to handle the positive and negative shift term. The term containing shifts lie on nodal points after discretization. Using finite difference formula the given differential equation was reduced to its conventional form and then transformed in to linear system of algebraic equations in the form of three term recurrence relations. The system of equations along with the boundary condition is solved by Gauss elimination method. To validate the applicability of the method some examples have been solved for different values of perturbation parameter and mesh size. The maximum absolute errors for the examples are calculated by using double mesh principle. And shows the effect of delay and advance parameters on the boundary layer behavior of the solution of the problem. The obtained numerical results have been compared with some of the previous findings reported in the literature and it is observed that the proposed method is more accurate than the previous findings reported in the literature.

The authors in [11] proposed exponentially fitted fourth-order finite difference approximation for singularly perturbed differential difference equation of delay and advance parameters. In this work the derivative of the given differential equation is replaced by finite difference approximation and transformed to leaner system of algebraic equations in the form of a three – term recurrence relations, which can easily be solved using a discrete invariant imbedding algorithm. Here, in order to validate the applicability of the proposed method some model examples have been solved for different values of the perturbation parameters and mesh size. The maximum absolute errors are computed using double mesh principle. Moreover, uses

uniform mesh technique over the domain. And investigates the influence of delay and advance parameters on the solution profile and, the stability and convergence of the scheme have been established. Computational rate of convergence for different values of perturbation parameter, delay and advance parameters and mesh sizes are tabulated for the considered numerical examples. The numerical result presented in the tables shows that the present method approximates the exact solution very well and depicts the betterment over some existing numerical methods reported in the literature.

The authors in [12]; Suggested, A Novel finite difference algorithm for solving singularly perturbed differential difference equation of second order with mixed shifts using a non-Polynomial spline with fitting factor. In this work, Taylor series expansion process is being used to produce an approximated form of the considered delay and advance terms in the governing differential equation. and changed in to fitted non – polynomial spline form. In order to discritize the solution domain it splits the interval in to N equal length sub- intervals with constant step size and then a fitted non – polynomial spline approach is devised in the form of a three – term recurrence relation. The convergence of the proposed method is examined and a quadratic rate of convergence is achieved. The maximum absolute error with quadratic rate of convergence of the solution recorded. The influence of delay and advance parameters investigated on the solution profile. To validate the applicability of the method some examples have been solved for different values of perturbation parameter and mesh size. The obtained numerical results have been compared with some of the previous findings reported in the literature and it is observed that the proposed method is more accurate than the previous findings reported in the literature. Thus, it is necessary to improve the accuracy of the solution with higher order of convergence.

Generally, from these presented works we observe that, most of the authors follow the uniform mesh style for the discretization of the solution domain, in order to adopt different finite difference methods for solving singularly perturbed differential equations with mixed shifts. However, the obtained accuracy of numerical solution takes attention to be work more. Hence, it is necessary to present accurate computational approach on piecewise mesh of the solution domain, which produces more accurate numerical solution for singularly perturbed differential equations with mixed shifts.

Chapter Three

Research Methodology

3.1 Study Site and Period

The study conducted in Jimma University under the department of Mathematics from January – October 2024.

3.2 Research Design

The study employed mixed design (Documentary review design and Experimental Design) on singularly perturbed differential equation with mixed shifts.

3.3 Source of Information

The relevant sources of information for this study were books, related published articles & related studies from internet and the experimental results obtained by writing MATLAB code for the present numerical methods

3.4 Mathematical Procedures

In order to achieve the stated specific objectives, the study followed the following procedures:

- Defining the singularly perturbed differential equations with mixed shifts.
- Discretizing the solution domain or interval by defining piecewise mesh widths
- Formulating an accurate computational approach.
- Establishing the convergence of the formulated method.
- Writing MATLAB code for the formulated method.
- Provide numerical illustrations to validate the applicability the approach and more accurateness.

Chapter Four

Description of the Computational Approach, and Results

4.1 Description of the problem

In this section, we consider the singularly perturbed differential equations with mixed shifts of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-\delta)+c(x)y(x)+d(x)y(x+\eta)=g(x),\quad x\in(0,1) \tag{4.1}$$

with the given interval conditions:

$$\begin{aligned} y(x)&=\phi(x),\quad x\in[-\delta,\,0],\\ y(x)&=\varphi(x),\quad x\in[1,\,1+\eta], \end{aligned} \tag{4.2}$$

where; ε $(0<\varepsilon<1)$ is perturbation parameter, and $\delta,\,\eta$ are left and right shift parameters respectively that satisfying $0<\delta,\,\eta<\varepsilon$. The functions $a(x),\ b(x),\ c(x),\ d(x),\ g(x)$ are assumed to be sufficiently smooth and bounded for guaranteeing the existence of unique continuous solutions. Since the delay and advance parameters respectively satisfy $0<\delta=o(\varepsilon)$ and $0<\eta=o(\varepsilon)$. The existence and uniqueness of a continuous solution for the problem defined by Eqs. (4.1) and (4.2) have been extensively analyzed in references [11, 12, 25, 26]. In this study, we concentrate on the computational approach that applied on a piecewise mesh in addressing numerical solution for singularly perturbed differential equations with mixed shift parameters.

Since $0<\delta,\eta<\varepsilon$, in order to manage the delay and advance parameter in the governing equation first we approximate the delay term $y(x-\delta)$ and advance term $y(x+\eta)$ using Taylor series expansion in the neighborhood of the point *x*, we have:

$$\begin{aligned} y(x-\delta)&\approx y(x)-\delta y'(x)+\frac{\delta^2}{2}y''(x)+O(\delta^3),\\ y(x+\eta)&\approx y(x)+\eta y'(x)+\frac{\eta^2}{2}y''(x)+O(\eta^3). \end{aligned} \tag{4.3}$$

On substituting Eq. (4.3) into Eq. (4.1), we get an asymptotically equivalent singularly perturbed differential equation:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)\left[y(x)-\delta y'(x)+\frac{\delta^2}{2}y''(x)\right]+c(x)y(x)+d(x)\left[y(x)+\eta y'(x)+\frac{\eta^2}{2}y''(x)\right]=g(x)$$

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)-\delta b(x)y'(x)+\frac{\delta^2}{2}b(x)y''(x)+c(x)y(x)+d(x)y(x)+\eta d(x)y'(x)+\frac{\eta^2}{2}d(x)y''(x)=g(x)$$

$$\left[\varepsilon+\frac{\delta^2}{2}b(x)+\frac{\eta^2}{2}d(x)\right]y''(x)+\left[a(x)-\delta b(x)+\eta d(x)\right]y'(x)+\left[b(x)+c(x)+d(x)\right]y(x)=g(x)$$

$$\varepsilon\left[1+\frac{\delta^2}{2\varepsilon}b(x)+\frac{\eta^2}{2\varepsilon}d(x)\right]y''(x)+\left[a(x)-\delta b(x)+\eta d(x)\right]y'(x)+\left[b(x)+c(x)+d(x)\right]y(x)=g(x)$$

Divide both sides by $1+\frac{1}{2\varepsilon}\left(\delta^2 b(x)+\eta^2 d(x)\right)$ yields:

$$y''(x)+\left[\frac{a(x)-\delta b(x)+\eta d(x)}{1+\frac{1}{2\varepsilon}\left(\delta^2 b(x)+\eta^2 d(x)\right)}\right]y'(x)+\left[\frac{b(x)+c(x)+d(x)}{1+\frac{1}{2\varepsilon}\left(\delta^2 b(x)+\eta^2 d(x)\right)}\right]y(x)=\frac{g(x)}{1+\frac{1}{2\varepsilon}\left(\delta^2 b(x)+\eta^2 d(x)\right)}$$

Now we can re-write this as asymptotically equivalent singular perturbation problem of the form:

$$\varepsilon y''(x)+p(x)y'(x)+q(x)y(x)=f(x),\quad x\in(0,1), \tag{4.4}$$

subject to the boundary condition:

$$\begin{aligned} y(0)&=\phi(0),\\ y(1)&=\varphi(1), \end{aligned} \tag{4.5}$$

where $p(x)=\dfrac{a(x)-\delta b(x)+\eta d(x)}{1+\dfrac{\delta^2 b(x)+\eta^2 d(x)}{2\varepsilon}}$, $q(x)=\dfrac{b(x)+c(x)+d(x)}{1+\dfrac{\delta^2 b(x)+\eta^2 d(x)}{2\varepsilon}}$, $f(x)=\dfrac{g(x)}{1+\dfrac{\delta^2 b(x)+\eta^2 d(x)}{2\varepsilon}}$.

Thus, the solution of Eqs. (4.4) and (4.5) is asymptotically equivalent to the solution of Eqs. (4.1) and (4.2). Generally, the solution of problems in Eqs. (4.4) and (4.5) exhibits boundary layer

behavior at one end of the interval [0, 1] depending on the sign of $p(x)$. Specifically, left layer exhaibits when $p(x) > 0$ and $q(x) \le 0$. While the problems whose solution exhibits layer behavior on the right side of the interval $[0, 1]$ for $p(x) < 0$ and $q(x) \ge 0$. Moreover, it is expected to be $p(x) \ne 0$ always in this defined problem.

4.2. **Piecewise Mesh Generation and Formulation of the Method**

We now construct a piecewise mesh that allocates a higher density of points in the layer regions compared to the non-layer regions. To differentiate between these regions, we introduce a transition parameter within the solution domain. This divides the domain into two subintervals with $[0, \tau], [\tau, 1]$ for the left layer or $[0, 1-\tau], [1-\tau, 1]$ for the right layer. The transition parameter is defined as:

$$\tau = \min\left\{\frac{1}{2},\ \varepsilon \ln(N)\right\}.. \tag{4.6}$$

This transition parameter denotes the width of the boundary layers. The domain $[0,1]^N$ is obtained by putting a non-uniform mesh with $\frac{N}{2}$ mesh elements in both inner and outer layer region. A general piecewise mesh $[0,1]^N = \{0 = x_0, x_1, x_2, ..., x_N = 1\}$ with step size $h_i = x_i - x_{i-1}$ which will be defined as

$$h_i = x_i - x_{i-1} = \begin{cases} \frac{2\tau}{N},\ i = 1, 2, ..., \frac{N}{2}, \\ \frac{2(1-\tau)}{N},\ i = \frac{N}{2} + 1, ..., N, \end{cases} \qquad \text{for left layer case} \tag{4.7}$$

Or

$$h_i = x_i - x_{i-1} = \begin{cases} \frac{2(1-\tau)}{N},\ \ i = 1, 2, ..., \frac{N}{2}, \\ \frac{2\tau}{N},\ \ \ \ i = \frac{N}{2} + 1, ..., N, \end{cases} \qquad \text{for right layer case} \tag{4.8}$$

where N is the number of mesh interval in the discretization. The target is to design a difference equation, which has the same general solution $y(x_i) \approx y_i$ as the differential equation in Eq (4.4) and Eq (4.5) at the mesh point (x_i)

Then, adapting from the Taylor series expansion, we approximate the approximate expansion:

$$y_{i+1} = y_i + h_{i+1}y_i' + \frac{h_{i+1}^2}{2}y_i'' + \frac{h_{i+1}^3}{3!}y_i''' + \frac{h_{i+1}^4}{4!}y_i^{(4)} + \ldots \tag{4.9}$$

$$y_{i-1} = y_i - h_i y_i' + \frac{h_i^2}{2}y_i'' - \frac{h_i^3}{3!}y_i''' + \frac{h_i^4}{4!}y_i^{(4)} + \ldots \tag{4.10}$$

From Eq. (4.9), using the forward finite difference approximation procedure, we obtain the second order finite difference approximation for the first derivative y_i' as:

$$y_i' = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{2}y_i'' + T_1 \tag{4.11}$$

with the truncation term, $T_1 = -\frac{h_{i+1}^2}{3!}y_i''' - \frac{h_{i+1}^3}{4!}y_i^{(4)} + \ldots \equiv O(h_{i+1}^2)$.

Similarly, adding Eqs. (4.9) and (4.10) (or applying central difference approximation) we obtain the second order finite difference approximation for the second derivative y_i'' as:

$$y_i'' = \frac{2}{h_i + h_{i+1}}\left[\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right] + T_2 \tag{4.12}$$

where $T_2 = -\left(\frac{h^2{}_i + h^2{}_{i+1}}{12}\right)y_i^{(4)} - \left(\frac{h^4{}_i + h^4{}_{i+1}}{360}\right)y_i^{(6)} + \ldots \equiv O(h_i^2 + h_{i+1}^2)$

Substituting Eqs. (4.11) and (4.12) in to the discrete form of Eq. (4.4), yields the finite difference approximation:

$$\varepsilon y_i'' + p_i y_i' + q_i y_i = f_i$$

$$\varepsilon y_i'' + p_i \left[\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{2} y_i'' \right] + q_i y_i + T_3 = f_i, \quad \text{for } T_3 = p_i T_1.$$

$$\left(\varepsilon - \frac{p_i h_{i+1}}{2} \right) y_i'' + p_i \left(\frac{y_{i+1} - y_i}{h_{i+1}} \right) + q_i y_i + T_3 = f_i$$

$$\left(\varepsilon - \frac{p_i h_{i+1}}{2} \right) \frac{2}{h_i + h_{i+1}} \left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \right) + \frac{p_i}{h_{i+1}} \left(y_{i+1} - y_i \right) + q_i y_i + T = f_i,$$

$$\left[\frac{2}{h_i (h_i + h_{i+1})} \left(\varepsilon - \frac{p_i h_{i+1}}{2} \right) \right] y_{i-1} + \left[\frac{-2}{h_i h_{i+1}} \left(\varepsilon - \frac{p_i h_{i+1}}{2} \right) - \frac{p_i}{h_{i+1}} + q_i \right] y_i + \left[\frac{2}{h_{i+1} (h_i + h_{i+1})} \left(\varepsilon - \frac{p_i h_{i+1}}{2} \right) + \frac{p_i}{h_{i+1}} \right] y_{i+1} = f_i$$

with truncation error: $T = \left(\varepsilon - \frac{p_i h_{i+1}}{2} \right) T_2 + T_3 \quad \cong \; O(h_i^2 + h_{i+1}^2) \equiv O(h^2)$.

This can be re-written in three- term recurrence relation form of the form:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1, \tag{4.13}$$

where

$$E_i = \frac{2}{h_i (h_i + h_{i+1})} (\varepsilon - \frac{p_i h_{i+1}}{2}), \qquad F_i = \frac{-2}{h_i h_{i+1}} (\varepsilon - \frac{p_i h_{i+1}}{2}) - \frac{p_i}{h_{i+1}} + q_i,$$

$$G_i = \frac{2}{h_{i+1} (h_i + h_{i+1})} (\varepsilon - \frac{p_i h_{i+1}}{2}) + \frac{p_i}{h_{i+1}}, \qquad H_i = f_i, \quad \text{and} \quad T \equiv O(h^2).$$

4.3. Thomas algorism

Using the Thomas Algorithm, to solve the obtained tri-diagonal system in Eq. (4.13), subject to the boundary conditions given in Eq. (4.5) that can be written as:

$$y_0 = y(0) = \phi(0) = \alpha, \quad \text{and} \quad y_1 = y(1) = \varphi(1) = \beta..$$

Assume that the solution of Eq. (4.13) can be written in the form of:

$$y_i = W_i y_{i+1} + T_i, \qquad i = N-1,\ N-2,\ \ldots\ 2,1 \tag{4.14}$$

where W_i and T_i are to be determined. Revisiting Eq. (4.14) at the nodal point x_{i-1}, we obtain:

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{4.15}$$

Substituting Eq. (4.15) into Eq. (4.13) that can be written as:

$$E_i \left\{ W_{i-1} y_i + T_{i-1} \right\} + F_1 y_i + G_i y_{i+1} = H_i ,$$

which leads to obtain the equation:

$$y_i = \frac{-G_i}{F_i + E_i W_{i-1}} y_{i+1} \quad + \quad \frac{H_i - E_i T_{i-1}}{F_i + E_i W_{i-1}} . \tag{4.16}$$

Comparing Eq. (4.14) with Eq. (4.16), we can determine:

$$\begin{cases} W_i = -\dfrac{G_i}{F_i + E_i W_{i-1}}, \\ T_i \ = \ \dfrac{H_i - E_i T_{i-1}}{F_i + E_i W_{i-1}}. \end{cases} \tag{4.17}$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1$, we need the initial conditions for $W_0 = 0$ and we take $T_0 = y_0$. With these stating point of initial values, we compute W_i and T_i for $i = 1, 2, \ldots, N-1$ from Eq. (4.13) in the forward process, and then obtain y_i in the backward

process from Eq. (4.14) and from the boundary condition $y_N = y(1)$. This procedure of solving system of equation is called as Thomas algorithm.

Further, to estimate the stability of the formulated scheme, we show that the formed coefficient matrix of the method satisfy the diagonal dominance. This diagonal dominance investigate by

$$|F_i| \geq |E_i| + |G_i|, \quad i = 1, 2, \ldots, N-1,$$

$$\left| \frac{-2}{h_i h_{i+1}} \left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) - \frac{p_i}{h_{i+1}} + q_i \right| \geq \left| \frac{2}{h_i (h_i + h_{i+1})} \left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) \right| + \left| \frac{2}{h_{i+1} (h_i + h_{i+1})} \left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) + \frac{p_i}{h_{i+1}} \right|,$$

$$\left| \frac{-2}{h_i h_{i+1}} \left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) + q_i \right| \geq \left| \frac{2}{h_i (h_i + h_{i+1})} \left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) \right| + \left| \frac{2}{h_{i+1} (h_i + h_{i+1})} \left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) \right|,$$

$$|q_i| \geq 0.$$

Thus, this implies that for each row, the sum of the two off-diagonal elements is less than or equal to the diagonal element. Therefore, the coefficient matrix of the obtained scheme is diagonally dominant. Henceforth, the Thomas algorithm is used for solving the formulated scheme in Eq. (4.13) is stable.

4.4 Improving Accuracy of the Solution (Richardson Extrapolation)

To enhance solution accuracy, we use Richardson extrapolation, a technique that improves the approximation by combining two solutions computed on different mesh sizes. By leveraging the known leading term in the truncation error, this method produces a higher-order approximation. Essentially, it accelerates convergence by forming a linear combination of two approximations on nested meshes, yielding a more accurate result.

Thus, since the formulated approximation method contains at Eqs. (4.13) which involves the truncation error of the formulated method is $O(h^2)$. Hence, we have an absolute error

$$\left| y(x_i) - y_i \right| \leq Ch^2, \tag{4.18}$$

where $y(x_i)$ and y_i are exact and approximate solutions respectively, C is constant independent of mesh sizes h_i. Let $[0,1]^{2N}$ be the mesh obtained by bisecting each mesh interval in $[0,1]^N$ and denote the approximation of the solution on $[0,1]^{2N}$ by $y_{\frac{i}{2}}$.

Consider Eq. (4.18) works for any $h_i \neq 0$, which implies:

$$y(x_i) - y_i \equiv Ch_i^2 + R^N, \quad x_i \in [0,1]^N \tag{4.19}$$

So that, it works for any $\frac{h_i}{2} \neq 0$ yields:

$$y(x_i) - y_{\frac{i}{2}} \equiv C\frac{h_i^2}{4} + R^{2N}, \quad x_i \in [0,1]^{2N}, \tag{4.20}$$

where the remainders, R^N and R^{2N} are $O(h^4) \equiv O(h_i^4 + h_{i+1}^4)$. Eliminating C from the system of equations formed by both Eqs. (4.19) and (4.20) leads to:

$$3y(x_i) - \left(4y_{\frac{i}{2}} - y_i \right) \equiv O(h^4), \tag{4.21}$$

which suggests that the extrapolated solution as:

$$\left(y_i \right)^{ext} = \frac{1}{3}\left(4y_{\frac{i}{2}} - y_i \right) \tag{4.22}$$

is also an approximation of $y(x_i)$. Using this approximation to evaluate the truncation error, we obtain:

$$\left| y(x_i) - \left(y_i \right)^{ext} \right| \leq Ch^4 \tag{4.23}$$

Using these two different solutions, which are obtained by the same method given by Eq.(4.13), we get another third solution in terms of the two by Eq. (4.22). This is Richardson extrapolation method for the formulated finite difference approximation only to accelerate the rate of convergence from second order to fourth order convergent.

4.5 Convergence Analysis

Writing the Tri-diagonal system of Eq. (4.13) in the matrix form as:

$$AY = V$$

In which; $A = \left(m_{ij}\right); i = 1,2,...N-1$ is a tri-diagonal matrix of order N-1.

Multiply both sides of Eq. (4.13) by $h_i h_{i+1}\left(h_i + h_{i+1}\right)$ we get

$$2h_{i+1}\left(\varepsilon - \frac{p_i h_{i+1}}{2}\right)y_{i-1} + \left[-2\left(h_i + h_{i+1}\right)\left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) - p_i h_i\left(h_i + h_{i+1}\right) + q_i h_i h_{i+1}\left(h_i + h_{i+1}\right)\right]y_i + \left[2h_i\left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) + p_i h_i\left(h_i + h_{i+1}\right)\right]y_{i+1} = f_i h_i h_{i+1}\left(h_i + h_{i+1}\right)$$

The system of this improved scheme is

$$\begin{cases}
\left[-2\left(h_1 + h_2\right)\left(\varepsilon - \frac{p_1 h_2}{2}\right) - p_1\left(h^2{}_1 + h_1 h_2\right) + q_1\left(h^2{}_1 h_2 + h_1 h^2{}_2\right)\right]y_1 + \left[2h_1\left(\varepsilon - \frac{p_1 h_2}{2}\right) + p_1\left(h^2{}_1 + h_1 h_2\right)\right]y_2 = f_1\left(h^2{}_1 h_2 + h_1 h^2{}_2\right) - 2h_2\left(\varepsilon - \frac{p_1 h_2}{2}\right)y_0;\ for i = 1 \\
2h_3\left(\varepsilon - \frac{p_2 h_3}{2}\right)y_1 + \left[-2\left(h_2 + h_3\right)\left(\varepsilon - \frac{p_2 h_3}{2}\right) - p_2\left(h^2{}_2 + h_2 h_3\right) + q_2\left(h^2{}_2 h_3 + h_2 h^2{}_3\right)\right]y_2 + \left[2h_2\left(\varepsilon - \frac{p_2 h_3}{2}\right) + p_2\left(h^2{}_2 + h_2 h_3\right)\right]y_3 = f_2\left(h^2{}_2 h_3 + h_2 h^2{}_3\right);\ for i = 2 \\
\vdots \\
\vdots \\
2h_{N-1}\left(\varepsilon - \frac{p_{N-2} h_{N-1}}{2}\right)y_{N-3} + \left[-2\left(h_{N-2} + h_{N-1}\right)\left(\varepsilon - \frac{p_{N-2} h_{N-1}}{2}\right) - p_{N-2}\left(h^2{}_{N-2} + h_{N-1} h_{N-2}\right) + q_{N-2}\left(h^2{}_{N-2} h_{N-1} + h_{N-2} h^2{}_{N-1}\right)\right]y_{N-2} + \\
\left[2h_{N-2}\left(\varepsilon - \frac{p_{N-2} h_{N-1}}{2}\right) + p_{N-2}\left(h^2{}_{N-2} + h_{N-1} h_{N-2}\right)\right]y_{N-1} = f_{N-2}\left(h^2{}_{N-2} h_{N-1} + h_{N-2} h^2{}_{N-1}\right);\ for i = N-2 \\
2h_N\left(\varepsilon - \frac{p_{N-1} h_N}{2}\right)y_{N-2} + \left[-2\left(h_{N-1} + h_N\right)\left(\varepsilon - \frac{p_{N-1} h_N}{2}\right) - p_{N-1}\left(h^2{}_{N-1} + h_{N-1} h_N\right) + q_{N-1}\left(h^2{}_{N-1} h_N + h_{N-1} h^2{}_N\right)\right]y_{N-1} = \\
f_{N-1}\left(h^2{}_{N-1} h_N + h_{N-1} h^2{}_N\right) - \left[2h_{N-1}\left(\varepsilon - \frac{p_{N-1} h_N}{2}\right) + p_{N-1}\left(h^2{}_{N-1} + h_{N-1} h_N\right)\right]y_N;\ for i = N-1
\end{cases}$$

The coefficient of matrix ‘A’ becomes

$$A=\begin{pmatrix} -2(h_1+h_2)\left(\varepsilon-\frac{p_1h_2}{2}\right)-p_1\left(h^2{}_1+h_1h_2\right)+q_1\left(h^2{}_1h_2+h_1h^2{}_2\right) & 2h_1\left(\varepsilon-\frac{p_1h_2}{2}\right)+p_1\left(h^2{}_1+h_1h_2\right) & 0 \\ 2h_3\left(\varepsilon-\frac{p_2h_3}{2}\right) & -2(h_2+h_3)\left(\varepsilon-\frac{p_2h_3}{2}\right)-p_2\left(h^2{}_2+h_2h_3\right)+q_2\left(h^2{}_2h_3+h_2h^2{}_3\right) & 2h_2\left(\varepsilon-\frac{p_2h_3}{2}\right)+p_2\left(h^2{}_2+h_2h_3\right) \\ 0 & \ddots & \ddots \\ 0 & \ddots & 2h_{N-2}\left(\varepsilon-\frac{p_{N-2}h_{N-1}}{2}\right)+p_{N-2}\left(h^2{}_{N-2}+h_{N-1}h_{N-2}\right) \\ 0 & \ddots & \ddots \\ 0 & 2h_N\left(\varepsilon-\frac{p_{N-1}h_N}{2}\right) & -2\left(h_{N-1}+h_N\right)\left(\varepsilon-\frac{p_{N-1}h_N}{2}\right)-p_{N-1}\left(h^2{}_{N-1}+h_{N-1}h_N\right)+q_{N-1}\left(h^2{}_{N-1}h_N+h_{N-1}h^2{}_N\right) \end{pmatrix}$$

Now we have

$$m_{ii+j}=2h_1\left(\varepsilon-\frac{p_1h_2}{2}\right)+p_1\left(h^2{}_1+h_1h_2\right)$$
$$m_{ii}=-2\left(h_1+h_2\right)\left(\varepsilon-\frac{p_1h_2}{2}\right)-p_1\left(h^2{}_1+h_1h_2\right)+q_1\left(h^2{}_1h_2+h_1h^2{}_2\right)$$
$$m_{ii-1}=2h_3\left(\varepsilon-\frac{p_2h_3}{2}\right)$$

And, $V=\left(d_i\right)$ is a column vector, where:

$$d_1=f_1\left(h^2{}_1h_2+h_1h^2{}_2\right)-2h_2\left(\varepsilon-\frac{p_1h_2}{2}\right)y_0$$
$$d_i=f_i\left(h^2{}_ih_{i+1}+h_ih^2{}_{i+1}\right);i=2,3,...,N-2$$
$$d_{N-1}=f_{N-1}\left(h^2{}_{N-1}h_N+h_{N-1}h^2{}_N\right)-\left(2h_{N-1}\left(\varepsilon-\frac{p_{N-1}h_N}{2}\right)+p_{N-1}\left(h^2{}_{N-1}+h_{N-1}h_N\right)\right)y_N$$

With local truncation error of

$$T_i\left(h\right)=C\left(h^4\right)\left(h_ih_{i+1}\left(h_i+h_{i+1}\right)\right);$$
$$T_i\left(h\right)=C\left(h^4\right)\left(h^2{}_ih_{i+1}+h_ih^2{}_{i+1}\right);C\left(h^4\right)=C\left(h^4{}_i+h^4{}_{i+1}\right)$$

C is constant

And, $Y=\left(y_1,y_2,y_3,...y_{N-1}\right)^t$ stands for approximate solution

Eq. (4.20) can also be written in error form as:

$$\begin{aligned} T_i(h) &= A\bar{Y} - V \\ V &= A\bar{Y} - T_i(h) \end{aligned} \tag{4.21}$$

Where, $\bar{Y} = \left(\bar{y}_1, \bar{y}_2, \bar{y}_3, ...\bar{y}_{N-1}\right)^t$ stands for the exact solution and

$T_i(h) = \left(T_1(h), T_2(h), T_3(h), ...T_{N-1}(h)\right)^t$ is local truncation error.

From Eq. (4.20) and Eq, (4.21) we have

$$\begin{aligned} AY &= V \\ AY &= A\bar{Y} - T_i(h) \\ T_i(h) &= A\bar{Y} - AY \\ A\left(\bar{Y} - Y\right) &= T_i(h) \end{aligned}$$

This implies that

$$AE = T(h)$$

Where: $E = \bar{Y} - Y = \left(e_1, e_2, e_3, ...e_{N-1}\right)^t$

Let S_i be the sum of elements of the ith row of matrix 'A', then we have

$$\begin{aligned} S_1 = \sum_{j=1}^{N-1} m_{ij} &= -2(h_1 + h_2)\left(\varepsilon - \frac{p_1 h_2}{2}\right) - p_1\left(h^2{}_1 + h_1 h_2\right) + q_1\left(h^2{}_1 h_2 + h_1 h^2{}_2\right) + 2h_1\left(\varepsilon - \frac{p_1 h_2}{2}\right) + p_1\left(h^2{}_1 + h_1 h_2\right) \\ &= -2(h_1 + h_2)\left(\frac{2\varepsilon - p_1 h_2}{2}\right) - p_1\left(h^2{}_1 + h_1 h_2\right) + q_1\left(h^2{}_1 h_2 + h_1 h^2{}_2\right) + 2h_1\left(\frac{2\varepsilon - p_1 h_2}{2}\right) + p_1\left(h^2{}_1 + h_1 h_2\right) \\ &= -2\varepsilon h_1 + p_1 h_1 h_2 - 2\varepsilon h_2 + p_1 h^2{}_2 - p_1 h^2{}_1 - p_1 h_1 h_2 + q_1\left(h^2{}_1 h_2 + h_1 h^2{}_2\right) + 2\varepsilon h_1 - p_1 h_1 h_2 + p_1 h^2{}_1 + p_1 h_1 h_2 \\ &= -2\varepsilon h_2 + p_1 h^2{}_2 + q_1\left(h^2{}_1 h_2 + h_1 h^2{}_2\right) \end{aligned}$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = 2h_{i+1}\left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) + \left[-2\left(h_i + h_{i+1}\right)\left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) - p_i\left(h^2{}_i + h_i h_{i+1}\right) + q_i\left(h^2{}_i h_{i+1} + h_i h^2{}_{i+1}\right)\right] + 2h_i\left(\varepsilon - \frac{p_i h_{i+1}}{2}\right) + p_i\left(h^2{}_{i+1} + h_i h_{i+1}\right)$$

$$= 2h_{i+1}\left(\frac{2\varepsilon - p_i h_{i+1}}{2}\right) + \left[-2\left(h_i + h_{i+1}\right)\left(\frac{2\varepsilon - p_i h_{i+1}}{2}\right) - p_i\left(h^2{}_i + h_i h_{i+1}\right) + q_i\left(h^2{}_i h_{i+1} + h_i h^2{}_{i+1}\right)\right] + 2h_i\left(\frac{2\varepsilon - p_i h_{i+1}}{2}\right) + p_i\left(h^2{}_{i+1} + h_i h_{i+1}\right)$$

$$= 2\varepsilon h_{i+1} - p_i h^2{}_{i+1} - 2\varepsilon h_i + p_i h_i h_{i+1} - 2\varepsilon h_{i+1} + p_i h^2{}_{i+1} + q_i\left(h^2{}_i h_{i+1} + h_i h^2{}_{i+1}\right) + 2\varepsilon h_i - p_i h_i h_{i+1}$$

$$= q_i\left(h^2{}_i h_{i+1} + h_i h^2{}_{i+1}\right); \quad \text{for } i = 2,3,\ldots N-2$$

$$S_{N-1} = \sum_{j=1}^{N-1} m_{ij} = 2h_N\left(\varepsilon - \frac{p_{N-1} h_N}{2}\right) + \left[-2\left(h_{N-1} + h_N\right)\left(\varepsilon - \frac{p_{N-1} h_N}{2}\right) - p_{N-1}\left(h^2{}_{N-1} + h_{N-1} h_N\right) + q_{N-1}\left(h^2{}_{N-1} h_N + h_{N-1} h^2{}_N\right)\right]$$

$$= 2h_N\left(\frac{2\varepsilon - p_{N-1} h_N}{2}\right) + \left[-2\left(h_{N-1} + h_N\right)\left(\frac{2\varepsilon - p_{N-1} h_N}{2}\right) - p_{N-1}\left(h^2{}_{N-1} + h_{N-1} h_N\right) + q_{N-1}\left(h^2{}_{N-1} h_N + h_{N-1} h^2{}_N\right)\right]$$

$$= 2\varepsilon h_N - p_{N-1} h^2{}_N - 2\varepsilon h_{N-1} + p_{N-1} h_{N-1} h_N - 2\varepsilon h_N + p_{N-1} h^2{}_N - p_{N-1} h^2{}_{N-1} - p_{N-1} h_{N-1} h_N + q_{N-1}\left(h^2{}_{N-1} h_N + h_{N-1} h^2{}_N\right)$$

$$= -2\varepsilon h_{N-1} - p_{N-1} h^2{}_{N-1} + q_{N-1}\left(h^2{}_{N-1} h_N + h_{N-1} h^2{}_N\right);$$

(4.23)

Now, we can choose 'h' sufficiently small, so that the matrix 'A' is irreducible and monotone, [24]. It follows that A^{-1} exist and its elements are non- negative. Hence, using Eq. (4.22) we have

$$\begin{aligned} &\mathrm{AE} = T(h) \\ &\mathrm{A}^{-1}(\mathrm{AE}) = \mathrm{A}^{-1}T(h) \\ &\mathrm{E} = \mathrm{A}^{-1}T(h) \end{aligned} \tag{4.24}$$

$$\text{And; } \|\mathrm{E}\| \le \left\|\mathrm{A}^{-1}\right\| \|T(h)\| \tag{4.25}$$

Let $\overline{m}_{k,i} \ge 0$ be the $(k,i)^{th}$ element of matrix A^{-1};

From the theory of matrices, we have

$$\sum_{i}^{N-1} \overline{m}_{k,i} S_i = 1; k = 1,2,3,\ldots,N-1 \tag{4.26}$$

Therefore

$$\sum_{i}^{N-1}\overline{m}_{k,i} \le \frac{1}{\min_{1\le i\le N-1} S_i} = \frac{1}{\|\beta_{io}\|\left(h^2{}_i h_{i+1} + h_i h^2{}_{i+1}\right)} \tag{2.27}$$

Where; $\beta_{io} = q_i$

We define; $\left\|\mathrm{A}^{-1}\right\| = 1 \le i \overset{Max}{\le} N-1 \sum_{i=1}^{N-1}\left|\overline{m}_{k,i}\right|$ and

$$\left\|T(h)\right\| = 1 \le i \overset{Max}{\le} N-1 \left|T_i(h)\right|$$

From Eq. (4.20), (4.24), (4.25) and Eq. (4.27) we observe that

$$e_i = \sum_{i}^{N-1}\overline{m}_{k.i} T_i(h); i = 1,2,3,...,N-1$$

This implies:

$$e_i \le \left(\sum_{i}^{N-1}\overline{m}_{k.i}\right) 1 \le i \overset{Max}{\le} N-1 \left|T_i(h)\right| \le \frac{C\left(h^4\right)\left(h^2{}_i h_{i+1} + h_i h^2{}_{i+1}\right)}{\|\beta_{io}\|\left(h^2{}_i h_{i+1} + h_i h^2{}_{i+1}\right)} \equiv C\left(h^4\right); \quad \text{C = Constant}$$

Where i_0 is some number between i and $N-1$

Therefore;

$$\|\mathrm{E}\| \equiv \mathrm{O}\left(h^4\right)$$

Hence, the proposed method reduced to fourth order convergent on piecewise mesh.

4.6 Numerical Examples and Results

To illustrate the applicability of the proposed method, we applied it to some modeled examples. For these considered examples where the exact solution is not known. So that, we use the double mesh principle to estimate the error and compute the experimental rate of convergence. Thus define the double mesh principle is given by:

$$D_{\varepsilon}^{N} = \max_{x_i \in [0,1]^N} | y_i^N - y_i^{2N} |,$$

where; y_i^N and y_i^{2N} respectively, denotes the numerical solution obtained using *N* and *2N* mesh intervals. Further, we calculate the order of convergence as:

$$R = \frac{\log(D_{\varepsilon}^{N}) - \log(D_{\varepsilon}^{2N})}{\log(2)}.$$

Example 1: Consider the singularly perturbed differential equation, [12, 25]

$$\varepsilon y''(x) + 2.5y'(x) - 2e^{x} y(x-\delta) - y(x) - xy(x+\eta) = 1,$$

subject to boundary constraints: $y(x) = \begin{cases} 1, & -\delta \le x \le 0, \\ 1, & 1 \le x \le 1+\eta. \end{cases}$

Example 2: Consider the singularly perturbed differential equation, [12, 25]

$$\varepsilon y''(x) - (1+e^{-x^2}) y'(x) - xy(x-\delta) - x^2 y(x) - (1.5-e^{-x}) y(x+\eta) = 1,$$

subject to boundary constraints: $y(x) = \begin{cases} 1, & -\delta \le x \le 0, \\ 1, & 1 \le x \le 1+\eta. \end{cases}$

Example 3: Consider the singularly perturbed differential equation, [11, 12]

$$\varepsilon y''(x) - (1+e^{x^2}) y'(x) - xy(x-\delta) + x^2 y(x) - (1.5-e^{-x}) y(x+\eta) = 1,$$

subject to boundary constraints: $y(x) = \begin{cases} 1, & -\delta \le x \le 0, \\ -1, & 1 \le x \le 1+\eta. \end{cases}$

Example 4: Consider the singularly perturbed differential equations, [26]

$$\varepsilon y''(x) - (1+e^{x^2})y'(x) - xy(x-\delta) + x^2 y(x) - (1-e^{-x})y(x+\eta) = 1,$$

subject to boundary constraints: $y(x) = \begin{cases} 1, & -\delta \le x \le 0, \\ -1, & 1 \le x \le 1+\eta. \end{cases}$

Table 1: Comparison of maximum absolute errors for Example 1when $\delta = \eta = 0.5\varepsilon$

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512	1024
Our Method						
2^{-5}	7.5384e-05	4.5374e-06	2.7985e-07	1.7431e-08	1.0886e-09	6.8513e-11
2^{-6}	1.5990e-04	1.4216e-05	1.1391e-06	6.3982e-08	3.5719e-09	2.1767e-10
2^{-7}	3.0706e-04	2.7248e-05	3.2356e-06	2.4805e-07	1.3546e-08	7.4884e-10
2^{-8}	5.4069e-04	3.3490e-05	5.8044e-06	6.5616e-07	4.9515e-08	2.6770e-09
2^{-9}	8.8420e-04	4.6736e-05	6.9499e-06	1.1258e-06	1.2499e-07	9.3852e-09
Results in [12]						
2^{-5}	5.108e-03	1.437e-03	3.712e-04	9.358e-05	2.344e-05	5.864e-06
2^{-6}	8.048e-03	2.759e-03	7.653e-04	1.968e-04	4.955e-05	1.241e-05
2^{-7}	9.448e-03	4.293e-03	1.438e-03	3.959e-04	1.016e-04	2.556e-05
2^{-8}	9.619e-03	5.019e-03	2.221e-03	7.353e-04	2.015e-04	5.165e-05
2^{-9}	9.641e-03	5.099e-03	2.590e-03	1.130e-03	3.718e-04	1.017e-04
Results in [25]						
2^{-5}	5.477e-03	1.526e-03	3.930e-04	9.902e-05	2.480e-05	6.204e-06
2^{-6}	8.487e-03	2.862e-03	7.898e-04	2.028e-04	5.105e-05	1.278e-05
2^{-7}	9.922e-03	4.413e-03	1.466e-03	4.024e-04	1.031e-04	2.596e-05
2^{-8}	1.009e-02	5.148e-03	2.252e-03	7.424e-04	2.032e-04	5.206e-05
2^{-9}	1.011e-02	5.228e-03	2.624e-03	1.138e-03	3.736e-04	1.021e-04

Table 2: Comparison of maximum absolute errors and rate of convergence for Example 1 when $\delta = \eta = 0.5\varepsilon$

$\varepsilon \downarrow N \rightarrow$	2^6	2^7	2^8	2^9	2^{10}
Present Method					
2^{-6}	1.4216e-05	1.1391e-06	6.3982e-08	3.5719e-09	2.1767e-10
	3.6415	4.1541	4.1629	4.0365	
2^{-10}	7.4278e-05	6.3072e-06	1.3201e-06	2.0858e-07	2.3014e-08
	3.5579	2.2564	2.6620	3.1800	
2^{-14}	3.2190e-04	1.8320e-05	1.1193e-06	1.2530e-07	3.0383e-08
	4.1351	4.0328	3.1591	2.0441	
2^{-18}	9.1026e-04	4.8968e-05	3.0952e-06	1.9215e-07	1.1556e-08
	4.0164	3.9837	4.0097	4.0055	
Results in [26]					
2^{-6}	4.2226e-03	1.2665e-03	5.4144e-04	2.6088e-04	1.2573e-04
	1.7372	1.2260	1.0534	1.0531	
2^{-10}	6.9397e-03	3.2824e-03	1.4610e-03	5.8199e-04	1.9090e-04
	1.0801	1.1678	1.3279	1.6081	
2^{-14}	7.5140e-03	3.7951e-03	1.8776e-03	9.2200e-04	4.4700e-04
	0.9854	1.0153	1.0260	1.0445	
2^{-18}	7.5337e-03	3.8215e-03	1.9255e-03	9.6516e-04	4.7804e-04
	0.9792	0.9889	0.9964	1.0136	

Table 3: Comparison of maximum absolute errors for Example 2 when $\delta = \eta = 0.5\varepsilon$

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512	1024
Our Method						
2^{-5}	5.6559e-05	3.3603e-06	2.1119e-07	1.3156e-08	8.2170e-10	5.1154e-11
2^{-6}	1.2504e-04	7.2009e-06	4.4130e-07	2.7445e-08	1.7186e-09	1.0428e-10
2^{-7}	2.4056e-04	1.3463e-05	8.1973e-07	5.1284e-08	3.1927e-09	2.0797e-10
2^{-8}	4.1777e-04	2.2639e-05	1.4198e-06	8.7975e-08	5.4846e-09	3.6030e-10
2^{-9}	6.7141e-04	3.6533e-05	2.2828e-06	1.4116e-07	8.8291e-09	5.4105e-10
Results in [12]						
2^{-5}	2.676e-03	6.223e-04	1.550e-04	3.864e-05	9.656e-06	2.413e-06
2^{-6}	4.201e-03	1.413e-03	3.222e-04	8.159e-05	2.024e-05	5.067e-06
2^{-7}	4.818e-03	2.205e-03	7.283e-04	1.642e-04	4.195e-05	1.041e-05
2^{-8}	4.980e-03	2.520e-03	1.132e-03	3.700e-04	8.300e-05	2.129e-05
2^{-9}	4.997e-03	2.604e-03	1.290e-03	5.736e-04	1.865e-04	4.172e-05
Results in [25]						
2^{-5}	1.858e-02	4.743e-03	1.190e-03	2.980e-04	7.452e-05	1.864e-05
2^{-6}	6.074e-02	1.988e-02	5.080e-03	1.275e-03	3.192e-04	7.981e-05
2^{-7}	1.111e-01	6.451e-02	2.061e-02	5.270e-03	1.323e-03	3.311e-04
2^{-8}	1.297e-01	1.176e-01	6.658e-02	2.101e-02	5.372e-03	1.349e-03
2^{-9}	1.310e-01	1.372e-01	1.212e-01	6.766e-02	2.122e-02	5.425e-03

Table 4: Comparison of maximum absolute errors for Example 3 when $\varepsilon = 0.1$

$N \rightarrow$	10^1	10^2	10^3	10^4
Our Method				
$\delta \downarrow$		$\eta = 0.5\varepsilon$		
0.00	7.5630e-03	1.3639e-05	1.3521e-09	6.7116e-10
0.05	7.8690e-03	1.4222e-05	1.4061e-09	6.2292e-10
0.09	9.4774e-03	1.7235e-05	1.6944e-09	6.4090e-10
$\eta \downarrow$		$\delta = 0.5\varepsilon$		
0.00	8.7742e-03	1.5858e-05	1.5637e-09	6.1925e-10
0.05	7.8690e-03	1.4222e-05	1.4061e-09	6.2292e-10
0.09	6.5342e-03	1.1839e-05	1.1777e-09	6.9480e-10
Results in [12]				
$\delta \downarrow$		$\eta = 0.5\varepsilon$		
0.00	8.123e-02	6.690e-03	3.573e-04	2.225e-05
0.05	8.066e-02	6.582e-03	3.529e-04	2.194e-05
0.09	8.019e-02	6.495e-03	3.494e-04	2.171e-05
$\eta \downarrow$		$\delta = 0.5\varepsilon$		
0.00	8.051e-02	6.527e-03	3.508e-04	2.180e-05
0.05	8.066e-02	6.582e-03	3.529e-04	2.194e-05
0.09	8.077e-02	6.626e-03	3.546e-04	2.206e-05
Results in [11]				
$\delta \downarrow$		$\eta = 0.5\varepsilon$		
0.00	9.109e-02	1.112e-02	6.382e-04	4.004e-05
0.05	9.047e-02	1.095e-02	6.306e-04	3.950e-05
0.09	8.996e-02	1.082e-02	6.244e-04	3.906e-05
$\eta \downarrow$		$\delta = 0.5\varepsilon$		
0.00	9.604e-02	1.116e-02	6.458e-04	3.924e-05
0.05	9.621e-02	1.124e-02	6.494e-04	3.950e-05
0.09	9.634e-02	1.131e-02	6.522e-04	3.970e-05

Table 5: Comparison of maximum absolute errors and rate of convergence for Example 4 when $\delta = \eta = 0.5\varepsilon$

$\varepsilon \downarrow N \rightarrow$	2^6	2^7	2^8	2^9	2^{10}
Present Method					
2^{-6}	9.7184e-06	5.9553e-07	3.7038e-08	2.3155e-09	1.4267e-10
	4.0285	4.0071	3.9996	4.0206	
2^{-10}	7.7403e-05	4.5880e-06	2.8862e-07	1.7978e-08	1.1235e-09
	4.0765	3.9906	4.0049	4.0002	
2^{-14}	3.2414e-04	1.8163e-05	1.1063e-06	6.9194e-08	4.3189e-09
	4.1575	4.0372	3.9990	4.0019	
2^{-18}	8.9748e-04	4.8811e-05	3.0521e-06	1.8957e-07	1.1456e-08
	4.0006	3.9993	4.0090	4.0486	

Results in [26]

2^{-6}	1.6341e-02	5.4210e-03	1.7037e-03	4.8628e-04	1.2196e-04
	1.5939	1.6767	1.8348	2.0692	
2^{-10}	1.6375e-02	5.4969e-03	1.7950e-03	5.6442e-04	1.7173e-04
	1.5749	1.6150	1.6694	1.7170	
2^{-14}	1.6339e-02	5.4848e-03	1.7947e-03	5.6613e-04	1.7459e-04
	1.5749	1.6112	1.6644	1.6975	
2^{-18}	1.6338e-02	5.4828e-03	1.7920e-03	5.6522e-04	1.7452e-04
	1.5756	1.6135	1.6647	1.6951	

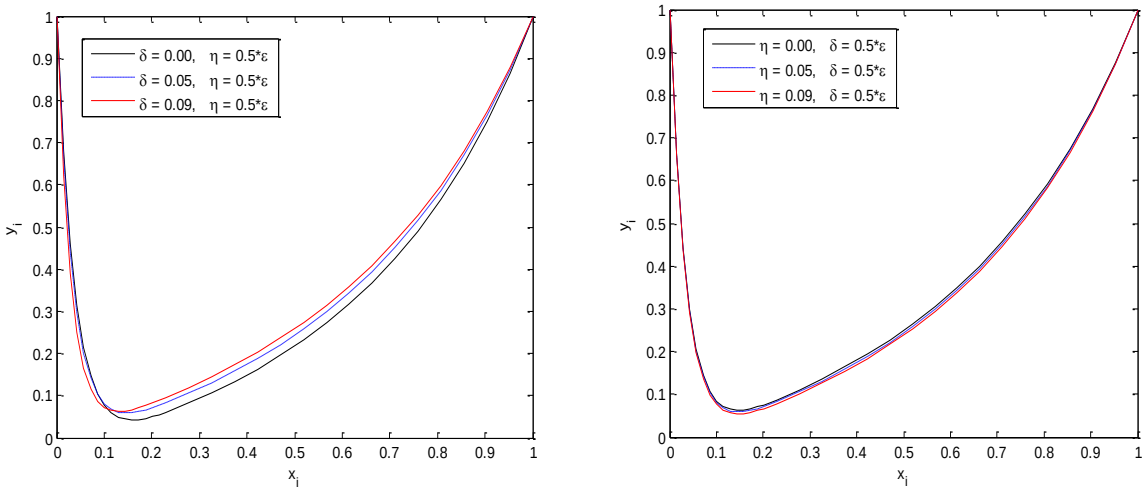

Figure 1: Layer profile in Examples 1 for $N = 32$, and $\varepsilon = 10^{-1}$.

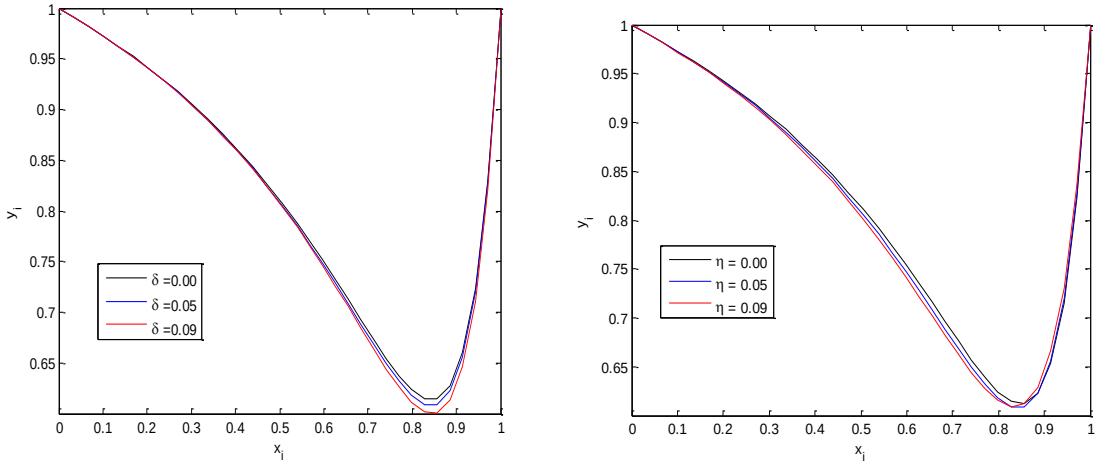

Figure 2: Layer profile in Example 2 with N= 32, and $\varepsilon = 10^{-1}$.

4.7 **Discussion**

This study presents an accurate computational method on a piecewise mesh for solving singularly perturbed differential equations with mixed shifts, which exhibit boundary layers at either end of the interval. First, the differential equation with the given interval conditions are transformed into asymptotically equivalent boundary value problems using Taylor series expansions for the delay and advance terms. Next, the solution domain is discretized piecewise based on the transition parameter, and the derivatives are replaced by finite difference approximations derived from Taylor expansions. As shown in Eq. (4.13), this results in a system of linear algebraic equations. Due to the diagonal dominance of this system, where the sum of off-diagonal elements is less than or equal to the diagonal element, it can be efficiently solved using the Thomas algorithm. To further improve accuracy, Richardson extrapolation is applied, enhancing the order of convergence from second to fourth.

To validate the proposed method, several modeled examples are provided, and the corresponding numerical results are presented in Tables 1–5. These tables offer a detailed comparison, demonstrating the accuracy of the solutions obtained. Figures 1 and 2 are used to illustrate the impact of the perturbation and shift parameters on the behavior of boundary layers, providing a visual representation of how these parameters influence the solution.

Additionally, the maximum absolute errors, as reported in Tables 1–5, clearly indicate that the solutions produced by the method are highly accurate. The convergence rates shown in Tables 2 and 6 further highlight the method's superiority over existing approaches in the literature. These results demonstrate that as the number of mesh points increases, the maximum absolute error decreases, confirming both the convergence and overall effectiveness of the method.

In summary, the numerical findings are consistent with the theoretical analysis and validate the main contribution of the presented method. The results confirm that the method achieves a high level of accuracy and performs better than previously established techniques.

Chapter Five

Conclusion and Future Work

5.1. Conclusion

This study introduces an efficient and accurate computational method for solving singularly perturbed differential equations with mixed shifts, particularly those exhibiting boundary layers. By transforming the differential equation into an asymptotically equivalent boundary value problem and utilizing a piecewise mesh based on the transition parameter, the method effectively captures the behavior of boundary layers. The finite difference approximations and the use of the Thomas algorithm for solving the resulting linear system ensure both accuracy and computational efficiency.

The application of Richardson extrapolation further enhances the method's accuracy by improving the convergence order from second to fourth. The numerical examples and results provided demonstrate the method's precision, as shown by the minimal maximum absolute errors and strong convergence rates. Additionally, visual representations of the perturbation and shift parameters' influence on boundary layers reinforce the effectiveness of the approach.

Overall, the method not only confirms the theoretical expectations but also outperforms previously established techniques. The proposed approach is highly effective for solving singularly perturbed problems, offering a reliable solution method with significant accuracy and convergence improvements.

5.2 Future Work

For future work, several avenues could be explored to further enhance and extend the presented method, such as: extension to higher-dimensional problems, application to nonlinear problems**,** and real-world applications.

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