Computational Approach on Piecewise Mesh for Singularly Perturbed Reaction Diffusion Problems



A Research Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics (Numerical Analysis)

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Declaration

I, the undersigned, hereby declare that this thesis, entitled **'Computational Approach on Piecewise Mesh for Singularly Perturbed Reaction Diffusion Problems**,' is my own original work. It has not been submitted for the award of any academic degree or similar qualification at any other institution or university. Furthermore, all sources used or quoted in this thesis have been fully indicated and acknowledged.

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Abstract

This study presents a computational approach using a piecewise mesh to solve singularly perturbed reaction-diffusion problems. The stability of the formulated scheme is thoroughly analyzed. Several numerical examples are provided to demonstrate the method's effectiveness. The results confirm that the method achieves high accuracy and efficiency, with reduced maximum absolute errors and improved convergence order. Additionally, the proposed approach outperforms several existing methods in terms of solution precision. Numerical experiments show enhanced convergence with fewer errors, supporting the theoretical analysis. As a result, the method proves to be highly effective in approximating exact solutions.

Chapter One
Introduction

1.1. Background of the Study

Advancements in computational mathematics have significantly enhanced the application of numerical methods across various fields, including medical science, engineering, and technology. Numerical analysis, a crucial branch of mathematics, focuses on developing and applying computational techniques to obtain approximate solutions for complex problems. These problems often include finding the roots of nonlinear equations, performing integration with intricate expressions, and solving differential equations when analytical solutions are elusive or impractical. Numerical methods become indispensable when exact solutions are difficult or impossible to obtain through direct analytical approaches. They provide practical ways to tackle real-world problems where traditional theoretical methods fall short. By offering approximate solutions with manageable computational effort, numerical analysis plays a pivotal role in addressing the challenges faced in applied sciences and engineering [1].

The problems that involve small positive parameters are known as perturbed problems, with the small parameter referred to as the perturbation parameter. These problems are classified into two categories based on the behavior of their solutions as the perturbation parameter approaches zero: regularly perturbed and singularly perturbed. If the solution of the perturbed problem approaches the solution of the reduced problem obtained by setting the perturbation parameter to zero, then the problem is classified as regularly perturbed. In such cases, asymptotic expansions are generally valid over the interval as the perturbation parameter tends to zero. Conversely, if the solution does not smoothly approach that of the reduced problem and exhibits different behavior due to the perturbation, the problem is classified as singularly perturbed. For singularly perturbed problems, asymptotic expansions are not uniformly valid over the interval, necessitating more specialized methods to accurately describe the solution [2].

A singularly perturbed differential equation is a differential equation in which the highest order derivative is multiplied by a small positive parameter. Due to the presence of this small perturbation parameter, the problem exhibits boundary or interior layers. When the order of the singularly perturbed differential equation is reduced by one, the problem is referred to as a

convection-diffusion type. If the order is reduced by two, the problem is known as a reaction-diffusion type, [3].

It is well-known that solutions to singular perturbation problems exhibit a multi-scale behavior. Specifically, there are thin transition layers, known as boundary layers, where the solution changes rapidly or experiences sudden jumps. In contrast, away from these layers, the solution behaves more regularly and varies slowly, referred to as the outer region. This multi-scale nature makes these problems challenging to address, as traditional numerical methods often become unstable and inaccurate for small perturbation values, [4]. Therefore, developing alternative numerical methods for solving singularly perturbed problems is crucial.

Singularly perturbed problems frequently arise in diverse fields of applied mathematics and engineering, such as fluid mechanics, elasticity, hydrodynamics, quantum mechanics, chemical-reactor theory, aerodynamics, plasma dynamics, rarefied-gas dynamics, oceanography, meteorology, semiconductor device modeling, diffraction theory, and reaction-diffusion processes, [5]. Due to the significant impact these problems have on real-world situations, developing numerical methods to approximate their solutions is of great value.

In numerical analysis, finite difference methods are used to solve differential equations by approximating derivatives with finite differences. Essentially, these methods transform differential equations into systems of algebraic equations that can be solved using iterative techniques. Traditionally, classical finite difference methods discretize the solution domain uniformly by defining a constant mesh width. However, for singularly perturbed problems, where a small positive parameter in the differential equation causes rapid changes in the solution within narrow regions, known as layers these classical methods can produce inaccurate results, [6, 7]. Therefore, it is crucial to develop finite difference methods that use piecewise meshes to accurately solve singularly perturbed reaction-diffusion problems.

Several numerical methods have been proposed by various authors for solving families of singularly perturbed reaction-diffusion problems. Many of these methods employ higher-order compact finite difference schemes (such as sixth, eighth, and tenth order) to address different types of singularly perturbed reaction-diffusion problems [8–10]. Traditionally, these higher-order finite difference methods are developed under the assumption that the coefficients of the

reaction term are constant. However, recent advances have introduced new approaches: in 2021 and 2023, respectively, a higher-order uniformly convergent numerical scheme and a fourth-order fitted mesh scheme were presented for singularly perturbed reaction-diffusion problems [11, 12]. These new methods have been shown to converge uniformly according to the discrete maximum norm, regardless of the perturbation parameter. Numerical results confirm the theoretical findings and demonstrate the validity and accuracy of these methods compared to existing techniques. Despite these advancements, most current finite difference methods are formulated on uniform meshes and assume constant coefficients for the reaction term. This limitation suggests that further work is needed to develop accurate solutions for problems where the reaction term coefficients are not constant.

Consequently, the primary drawbacks of most of the methods presented include the reliance on a uniform mesh within the solution domain and the restriction of the coefficient reaction term to a constant function. These methods typically yield accurate solutions only when the mesh size and the perturbation parameter values are comparable. Therefore, the main objective of this work is to introduce a computational finite difference approach on piecewise mesh of the solution domain for addressing singularly perturbed problems, particularly when the coefficient of the reaction term is variable.

1.2 Statement of the Problem

Various numerical methods have been used to tackle singularly perturbed reaction- diffusion problems. However, these methods often require further refinement to enhance the accuracy of numerical solutions. To address this, we use a piecewise mesh, which has the potential to deliver more precise numerical results for these types of problems.

Thus, this study aims to address the following key research questions:

- How to formulate a computational approach on piecewise mesh to solve the singularly perturbed reaction-diffusion problems?
- How to investigate the stability analysis for the formulated approach?
- To what extent the accuracy will be yields by the proposed approach, is well advantageous than the existing methods?

1.3 Objectives of the Study

1.3.1 General objective

The general objective of this study is to introduce a computational approach on piecewise mesh for solving singularly perturbed reaction diffusion problems.

1.3.2 Specific objectives

The specific objectives of the study are:

- To formulate a computational approach on piecewise mesh for solving singularly perturbed reaction diffusion problems.
- To provide the stability analysis for the formulated numerical approach.
- To investigate the more accuracies' of the proposed approach.

1.4 Delimitation of the Study

This study is delimited to study a computational approach on piecewise mesh for singularly perturbed reaction-diffusion problems of the form:

$$-\varepsilon y''(x) + b(x)y(x) = f(x), \qquad \forall x \in (0,1), \tag{1.1}$$

subject to the boundary conditions:

$$y(0) = \eta_1 \quad \text{and} \quad y(1) = \eta_2 \tag{1.2}$$

where ε $(0 < \varepsilon << 1)$ is perturbation parameter, and η_1, η_2 are given constants with the unknown function $y(x)$. Assume that the functions $f(x)$ and $b(x)$ are sufficiently smooth and $b(x) \geq b_0 > 0$ for some constant b_0. As the problem is of the reaction-diffusion type, the condition $b(x) \geq b_0 > 0$ ensure the existence of dual boundary layers near the two end points of the solution domain.

1.5 Significance of the Study

The outcomes of this study:

- Produce a more accurate numerical solution for the considered problems.
- Provide a reference material for scholars to develop higher order numerical approach for solving singularly perturbed reaction diffusion problems.
- Help the graduate students to acquire research skills and scientific procedures.

Chapter Two
Review of Related Literature

2.1 Singularly Perturbed Problems and Numerical Methods

In applied science and engineering, many physical and biological problems involve solutions characterized by boundary and interior layers, depending on specific parameter values. These problems are known as singular perturbation problems (SPPs). In such problems, no single asymptotic expansion is universally valid across the entire interval as the perturbation parameter approaches zero. The presence of boundary or interior layers complicates both numerical and analytical treatments, as these layers create regions where the solution changes rapidly, while outside these regions, the solution varies gradually and uniformly. As the perturbation parameter approaches zero, the regularity of the solution tends to decrease significantly [4, 6, 7].

Perturbation theory explores the effects of small parameters on mathematical models, particularly in ordinary differential equations. In singular perturbation problems, the influence of a small parameter cannot be approximated by simply setting it to zero. The concept of singular perturbation was first introduced by Prandtl in 1904 during a presentation on fluid motion with small friction at the Third International Congress of Mathematicians in Heidelberg. In his report, Prandtl demonstrated that fluid flow over a body could be divided into two regions: the boundary layer and the outer region. The term ‘singular perturbations’ was first coined by Friedrichs in a paper presented at a seminar on nonlinear vibrations at New York University [8-10].

Solutions to singular perturbation problems typically exhibit layer behavior, where sharp changes occur in narrow regions known as boundary or interior layers. Although Prandtl introduced the term ‘boundary layer,’ it became widely recognized following subsequent research. Many theoretical and applied problems in science and technology lead to boundary value problems involving singularly perturbed differential equations with a multi-scale character. However, these problems are often too complex to be fully resolved analytically, necessitating the use of numerical simulations to gain valuable insights [11, 12].

Singularly perturbed problems arise in the modeling of various complex processes, including fluid flow at high Reynolds numbers, water quality issues in river networks, convective heat transport, semiconductor device modeling, electromagnetic field problems in moving media, financial modeling for option pricing, turbulence models, simulations of oil extraction, the theory of plates and shells, atmospheric pollution, groundwater transport, and chemical reactor dynamics [13-17].

It is well-known that the solutions to singularly perturbed problems exhibit sharp boundary or interior layers when the perturbation parameter is very small, meaning that the solution changes rapidly in specific regions while remaining relatively stable elsewhere [6, 7]. These challenges make accurate numerical solutions difficult to achieve, requiring advanced computational techniques to manage them effectively [17, 18]. Therefore, advanced computational techniques are essential for effectively solving these problems.

2.2 Recent Works on Singularly Perturbed Reaction-Diffusion Problems

The work referenced in [16 - 19] presents a numerical methods utilizing a finite difference scheme with a uniform mesh to solve singularly perturbed two-point boundary value problems for 1D reaction-diffusion equations. Initially, the differential equation's derivatives are replaced by finite difference approximations. The problem is then solved using a fourth-order compact finite difference method on a uniform mesh. The method's efficiency is demonstrated through numerical illustrations, supported by graphs that reinforce the numerical results. The theoretical and computational rates of convergence have been analyzed and show good agreement. As evident from the numerical results in the tables and graphs, the proposed method closely approximates the exact solution.

The authors in [8 – 12, 16 - 21] introduced different family of the finite difference methods to solve one-dimensional singular reaction-diffusion problems. The method involves replacing the differential equation's derivatives with finite difference approximations. The resulting difference equation is then transformed into a system of linear algebraic equations in the form of a three-term recurrence relation, which can be efficiently solved using a discrete invariant imbedding

algorithm. To demonstrate the method's effectiveness, several model examples were solved using various perturbation parameters and mesh sizes. Theoretical error bounds and the numerical rate of convergence were also established. The numerical results, displayed in tables and graphs, indicate that the proposed method closely approximates the exact solution.

From the methods presented, it's evident that most authors use a uniform mesh for discretizing the solution domain. However, greater emphasis on solution accuracy is necessary. Therefore, in our study, we will employ a piecewise mesh approach to solve singularly perturbed differential equations with a reaction term, as this method is expected to yield a more accurate solution compared to those previously discussed.

Chapter Three

Research Methodology

3.1 Study Site and Period

The study conducted at Jimma University under College of Natural Science, Department of Mathematics in numerical analysis stream from June – December, 2024.

3.2 Study Design

The study employed both documentary review and experimental design on a computational approach on piecewise mesh for solving singularly perturbed reaction-diffusion problems.

3.3 Sources of Information

The relevant sources of information for this study were journals, books, published articles, and related articles from internet.

3.4 Mathematical procedures

In order to achieve the stated specific objectives, the study followed the following procedures:

1. Defining the problem.
2. Discretizing the solution domain or interval by defining piecewise mesh widths.
3. Formulating Numerical Scheme on Piecewise Mesh
4. Establishing the stability of the formulated scheme.
5. Writing MATLAB code for the formulated scheme.
6. Provide numerical illustrations to validate the applicability the approach and more accurateness.

Chapter Four

Description of the Computational Approach, and Results

4.1 Description of the problem

In this section, the description of the method has been presented for solving a class of singularly perturbed reaction-diffusion problems of the form:

$$-\varepsilon y''(x) + b(x)y(x) = f(x), \qquad \forall x \in (0,1), \tag{4.1}$$

subject to the boundary conditions:

$$y(0) = \eta_1 \quad \text{and} \quad y(1) = \eta_2 \tag{4.2}$$

where ε $(0 < \varepsilon << 1)$ is perturbation parameter, and η_1, η_2 are given constants with the unknown function $y(x)$. Assume that the functions $f(x)$ and $b(x)$ are sufficiently smooth and assumed to their higher order derivatives are exist with $b(x) \geq b_0 > 0$ for some constant b_0. As the problem is of the reaction-diffusion type, the condition $b(x) \geq b_0 > 0$ ensure the existence of dual boundary layers near the two end points of the solution domain.

The existence and uniqueness of a continuous solution for the problem defined by Eqs. (4.1) and (4.2) have been extensively analyzed in references [10, 11]. In this study, we concentrate on the computational methodology applied to a piecewise mesh in addressing numerical solution for singularly perturbed reaction-diffusion problems.

4.2 Piecewise Mesh Generation

We construct a non-uniform mesh that contains more points in the layer regions than non-layer region. Given the nature of the problem, which involves both left and right boundary layers, the domain is partitioned into three sub-intervals. The subdivided domain $[0,1]^N$, $N \geq 4$ is divided into three subintervals, $[0,\tau]$, $[\tau,1-\tau]$, $[1-\tau,1]$ where the chosen transition parameter:

$$\tau = \min\left\{\frac{1}{4},\ 4\sqrt{\varepsilon}\ln(N)\right\}, \tag{4.3}$$

related with the width of the boundary layers, [6, 7].

The domain $[0,1]^N$ is obtained by putting a non-uniform mesh with $\frac{N}{4}$ mesh elements in both the layer intervals and a uniform mesh with $\frac{N}{2}$ mesh elements in the outer layer region. A general non-uniform mesh $[0,1]^N = \{0 = x_0, x_1, x_2, ..., x_N = 1\}$ with step size will be defined as

$$h_i = x_i - x_{i-1} = \begin{cases} \frac{4\tau}{N},\ i = 1, 2, ..., \frac{N}{4} \\ \frac{2(1-2\tau)}{N},\ i = \frac{N}{4}+1, ..., \frac{3N}{4}, \\ \frac{4\tau}{N},\ i = \frac{3N}{4}+1, ..., N. \end{cases} \tag{4.4}$$

4.3 Formulation of the Computational Approach

Now, in order to develop the finite difference approximation for the differential equation in Eq. (4.1), the interval $[0, 1]$ is divided into N sub-intervals. For this, we introduce the set of grid points:

$$x_i = x_{i-1} + h_i, \quad \begin{cases} i = 1, 2, \ldots, N, \\ h_i = x_i - x_{i-1}, \\ 0 = x_0 < x_1 < \ldots < x_N = 1. \end{cases} \tag{4.5}$$

Assume that $y(x_i)$ be the exact solution of the Eqs. (4.1) and (4.2) and y_i be an approximation to $y(x_i)$. Also, adopt that $y(x)$ has continuous higher order derivatives on $[0, 1]$, and then to find a description of the finite difference approximation, we use Taylor's series expansion in order to get the formula for y_i'' as:

$$y_{i+1} = y_i + h_{i+1}y_i' + \frac{h_{i+1}^2}{2!}y_i'' + \frac{h_{i+1}^3}{3!}y_i''' + \frac{h_{i+1}^4}{4!}y_i^{(4)} + \frac{h_{i+1}^5}{5!}y_i^{(5)} + \frac{h_{i+1}^6}{6!}y_i^{(6)} + \ldots \quad (4.6)$$

$$y_{i-1} = y_i - h_i y_i' + \frac{h_i^2}{2!}y_i'' - \frac{h_i^3}{3!}y_i''' + \frac{h_i^4}{4!}y_i^{(4)} - \frac{h_i^5}{5!}y_i^{(5)} + \frac{h_{i+1}^6}{6!}y_i^{(6)} + \ldots \quad (4.7)$$

Multiplying Eq. (4.6) by h_i and Eq. (4.7) by h_{i+1}, and then adding the two equation yields:

$$\begin{aligned} h_i y_{i+1} + h_{i+1}y_{i-1} = h_i y_i + h_{i+1}y_i + h_i h_{i+1}(\frac{h_{i+1}+h_i}{2})y_i'' + h_i h_{i+1}(\frac{h_{i+1}^2 - h_i^2}{6})y_i''' + \\ h_i h_{i+1}(\frac{h_{i+1}^3 + h_i^3}{24})y_i^{(4)} + h_i h_{i+1}(\frac{h_{i+1}^4 - h_i^4}{120})y_i^{(5)} + \\ h_i h_{i+1}(\frac{h_{i+1}^5 + h_i^5}{720})y_i^{(6)} + \ldots \end{aligned}$$

From this equation, let's solve for y_i'', and then rearrange it into the form:

$$y_i'' = \frac{2}{h_{i+1}+h_i}(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}) - (\frac{h_{i+1}-h_i}{3})y_i''' - (\frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)})y_i^{(4)} + \tau_1, \quad (4.8)$$

with the considered truncation term is

$$\tau_1 = -(\frac{h_{i+1}^4 - h_i^4}{60(h_{i+1}+h_i)})y_i^{(5)} - (\frac{h_{i+1}^5 + h_i^5}{360(h_{i+1}+h_i)})y_i^{(6)} + \ldots$$

Now, substituting Eq. (4.8) into the discrete form of Eq. (4.1) gives:

$$\frac{-2\varepsilon}{h_{i+1}+h_i}(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}) + \varepsilon(\frac{h_{i+1}-h_i}{3})y_i''' + \varepsilon(\frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)})y_i^{(4)} + b_i y_i + \tau_2 = f_i, \quad (4.9)$$

where $\tau_2 = -\varepsilon\tau_1$.

Differentiating Eq. (4.1) twice and solving for y_i''', $y_i^{(4)}$ at the discrete point, we get:

$$y_i''' = \frac{b_i y_i' + b_i' y_i - f_i'}{\varepsilon},$$
$$y_i^{(4)} = \frac{b_i y_i'' + 2b_i' y_i' + b_i'' y_i - f_i''}{\varepsilon}. \tag{4.10}$$

Also, substituting Eq. (4.10) into Eq. (4.9), yields:

$$\begin{aligned} \frac{-2\varepsilon}{h_{i+1}+h_i}\left(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) + \left[b_i + b_i'\left(\frac{h_{i+1}-h_i}{3}\right) + b_i''\left(\frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)}\right)\right] y_i \\ + \left[b_i\left(\frac{h_{i+1}-h_i}{3}\right) + \left(b_i' \frac{h_{i+1}^3 + h_i^3}{6(h_{i+1}+h_i)}\right)\right] y_i' \\ + \left(b_i \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)}\right) y_i'' = f_i + \left(\frac{h_{i+1}-h_i}{3}\right) f_i' + \left(\frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)}\right) f_i'' \end{aligned} \tag{4.11}$$

Using Eqs. (4.6) and (4.7), we have the finite difference approximation:

$$y_i' = \frac{y_{i+1}-y_i}{h_{i+1}} + \tau_3,$$
$$y_i'' = \frac{2}{h_{i+1}+h_i}\left(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) + \tau_4, \tag{4.12}$$

where the local truncation errors are:

$$\tau_3 = -\frac{h_{i+1}}{2} y_i'', \quad \text{and} \quad \tau_4 = -\left(\frac{h_{i+1}-h_i}{3}\right) y_i'''.$$

Finally, substituting Eq. (4.12) in to Eq. (4.11) and after rearranging, we obtain the three-term recurrence relations scheme of the form:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1, \tag{4.13}$$

where

$$E_i = \frac{2}{h_i(h_{i+1}+h_i)}[-\varepsilon + b_i \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)}],$$

$$F_i = \frac{2}{h_{i+1}h_i}[\varepsilon - b_i \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)}] + b_i + b_i'(\frac{h_{i+1}-h_i}{3}) + b_i''(\frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)}) - \frac{1}{h_{i+1}}[b_i(\frac{h_{i+1}-h_i}{3}) + (b_i' \frac{h_{i+1}^3 + h_i^3}{6(h_{i+1}+h_i)})],$$

$$G_i = \frac{2}{h_{i+1}(h_{i+1}+h_i)}[-\varepsilon + b_i \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)}] + \frac{1}{h_{i+1}}[b_i(\frac{h_{i+1}-h_i}{3}) + (b_i' \frac{h_{i+1}^3 + h_i^3}{6(h_{i+1}+h_i)})],$$

$$H_i = f_i + (\frac{h_{i+1}-h_i}{3}) f_i' + (\frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)}) f_i''.$$

4.4 Improving Accuracy of the Solution (Richardson extrapolation)

In order to improve the accuracy of the solution, we apply Richardson extrapolation technique. The basic idea behind extrapolation is that whenever the leading term in the truncation error for an approximation formula is known, we can combine two approximations obtained from that formula using different values of the mesh sizes *h* and $\frac{h}{2}$ to obtain a higher-order approximation and the technique is known as Richardson extrapolation. This procedure is a convergence acceleration technique which consists of considering a linear combination of two computed approximations of a solution (on two nested meshes). The linear combination turns out to be a better approximation.

Hence, while deriving the finite difference approximations in Eqs. (4.8) and (4.12), the truncation term denoted by:

$$h^2 \approx \frac{h_{i+1}-h_i}{3} b_i \frac{h_{i+1}}{2} y_i'' + \ldots,$$

and then the formulated approximation method contains at Eq. (4.13) which involves the truncation error of the formulated method is $O(h^2)$. Hence, we have

$$|y(x_i) - y_i| \leq Ch^2 \qquad (4.14)$$

where $y(x_i)$ and y_i are exact and approximate solutions respectively, C is constant independent of mesh sizes h. Let $[0,1]^{2N}$ be the mesh obtained by bisecting each mesh interval in $[0,1]^N$ and denote the approximation of the solution on $[0,1]^{2N}$ by $y_{\frac{i}{2}}$. Consider Eq. (4.14) works for any $h \neq 0$, which implies:

$$y(x_i) - y_i \cong Ch^2 + R^N, \qquad x_i \in [0,1]^N \tag{4.15}$$

So that, it works for any $\frac{h}{2} \neq 0$ yields:

$$y(x_i) - y_{\frac{i}{2}} \cong C\frac{h^2}{4} + R^{2N}, \qquad x_i \in [0,1]^{2N} \tag{4.16}$$

where the remainders, R^N and R^{2N} are $O(h^4)$. Eliminating C from the system of equations formed by both Eqs. (4.15) and (4.16) leaders to:

$$3y(x_i) - \left(4y_{\frac{i}{2}} - y_i\right) \cong O(h^4), \tag{4.17}$$

which suggests that the extrapolated solution as:

$$\left(y_i\right)^{ext} = \frac{1}{3}\left(4y_{\frac{i}{2}} - y_i\right) \tag{4.18}$$

is also an approximation of $y(x_i)$. Using this approximation to evaluate the truncation error, we obtain:

$$\left|y(x_i) - \left(y_i\right)^{ext}\right| \leq Ch^4 \tag{4.19}$$

Now, using these two different solutions which are obtained by the same method given by Eq. (4.13), we get another third solution in terms of the two by Eq. (4.18). This is Richardson extrapolation method for the formulated finite difference approximation only to accelerate the rate of convergence from second to fourth order convergent method.

4.5 Thomas Algorithm

In this section, a brief description for solving the tri-diagonal system using the discrete invariant imbedding algorithm, also called the Thomas Algorithm, is presented as follows. Consider the method in Eq. (4.13), for $i = 1, 2, \ldots, N-1$ and subject to the boundary conditions given in Eq. (4.2) that can be written as:

$$y(0) = y_0 = \eta_1, \text{ and } \quad y(1) = y_N = \eta_2.$$

Assume that the solution of Eq. (4.13) can be written as:

$$y_i = W_i y_{i+1} + T_i, \qquad i = N-1,\ N-2,\ \ldots\ 2,1 \tag{4.20}$$

where W_i and T_i are to be determined.

By re-considering Eq. (4.20) at the nodal point x_{i-1}, we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{4.21}$$

Substituting Eq. (4.21) into Eq. (4.13) that can be written as:

$$E_i \left\{ W_{i-1} y_i + T_{i-1} \right\} + F_1 y_i + G_i y_{i+1} = H_i,$$

which leads to obtain the equation:

$$y_i = \frac{-G_i}{F_i + E_i W_{i-1}} y_{i+1} \quad + \quad \frac{H_i - E_i T_{i-1}}{F_i + E_i W_{i-1}}. \tag{4.22}$$

Comparing Eq. (4.20) with Eq. (4.22), we can determine:

$$\begin{cases} W_i = -\dfrac{G_i}{F_i + E_i W_{i-1}}, \\ T_i = \dfrac{H_i - E_i T_{i-1}}{F_i + E_i W_{i-1}}. \end{cases} \tag{4.23}$$

To solve these recurrence relations for $i = 1, 2, ..., N-1$, we need the initial conditions for $W_0 = 0$ and we take $T_0 = y_0 = \eta_1$. With these stating point of initial values, we compute W_i and T_i for $i = 1, 2, ..., N-1$ from Eq. (4.13) in the forward process, and then obtain y_i in the backward process from Eq. (4.20) and from the boundary condition $y_N = \eta_2$. This procedure of solving system of equation is called as Thomas algorithm.

4.6 Stability

The formulation scheme, in Eq. (4.13) constitute the system of $(N-1)\times(N-1)$ linear algebraic equations, that give the approximations for $y_1, \ldots, y_{N-1}$ of the solution $y(x)$ at the nodal points, $x_1, \ldots, x_{N-1}$, with piecewise mesh length h_i. This can be re-written in matrix form

$$MY = R, \tag{4.24}$$

where the three matrices are defined as:

$$M = (m_{ij})_{(N-1)\times(N-1)} = \begin{bmatrix} F_1 & G_1 & 0 & \cdots & 0 \\ E_2 & F_2 & G_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & E_{N-2} & F_{N-2} & G_{N-2} \\ 0 & \cdots & 0 & E_{N-1} & F_{N-1} \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix}, \quad R = \begin{bmatrix} H_1 - E_0 y_0 \\ H_2 \\ \vdots \\ H_{N-2} \\ H_{N-1} - G_N y_N \end{bmatrix},$$

Theorem 1: (Stability)

Let M be a coefficient matrix of the tri-diagonal system, in Eq. (4.24). Then, the matrix M is an irreducible and diagonally dominant matrix and hence the scheme is stable, [22].

Proof: To estimate the stability of the formulated method, we show that the coefficient matrix of the method satisfy the diagonal dominance. First, to check this, consider that $b(x) \geq b_0 > 0, \ \forall x \in [0, 1]$ and for $i = 1, \ldots, N-1$, from Eq. (4.16), we have:

$$F_i \geq E_i + G_i, \quad i = 1, 2, \ \ldots, N-1,$$

Thus, this implies that for each row of M, the sum of the two off-diagonal elements is less than or equal to the diagonal element. Thus, M is diagonally dominant. Hence, M is irreducible matrix. Henceforth, the Thomas algorithm is used for solving the formulated scheme in Eq. (4.24) is stable for the described scheme. Therefore, the consistency criteria had shown using the truncation error in Eqs. (4.8) and (4.12), and the stability theorem above, it is guaranteed that the convergence of the formulated scheme.

4.7 Numerical Examples and Results

In order to test the validity of the proposed method and to demonstrate their convergence computationally, we have taken different model examples of a linear second order two point boundary value problems with exact solutions. The pointwise absolute errors (*AE*) at the nodal points are given by:

$$AE_N = \max_{1 \leq i \leq N-1} | y(x_i) - (y_i)^{ext} |.$$

where $y(x_i)$ and $\left(y_i\right)^{ext}$ are exact solution and numerical solution obtained by Richardson extrapolation respectively, at the nodal point x_i. Further, we calculate the order of convergence as:

$$R = \frac{\log(AE_N) - \log(AE_{2N})}{\log(2)}.$$

Example 1: Consider the singularly perturbed problem, [12]:

$$\begin{cases} -\varepsilon y''(x) + (1 + x(1-x))y(x) = f(x), \quad 0 < x < 1, \\ \\ y(0) = 0 = y(1). \end{cases}$$

.

The source function is given by:

$$f(x) = 1 + x(1-x) + (2\sqrt{\varepsilon} - x^2(1+x))\exp(\frac{-(1-x)}{\sqrt{\varepsilon}}) + (2\sqrt{\varepsilon} - x(1-x)^2)\exp(\frac{-x}{\sqrt{\varepsilon}}).$$

The exact solution is $y(x) = 1 + (x-1)\exp(\frac{-x}{\sqrt{\varepsilon}}) - x\exp(\frac{-(1-x)}{\sqrt{\varepsilon}})$.

Example 2: Consider the singularly perturbed problem, [11]:

$$\begin{cases} -\varepsilon y''(x) + y(x) = 1 + 2\sqrt{\varepsilon}(\exp(\frac{x}{\sqrt{\varepsilon}}) + \exp(\frac{x-1}{\sqrt{\varepsilon}})), \quad 0 < x < 1, \\ \\ y(0) = 0 = y(1). \end{cases}$$

The exact solution is $y(x) = 1 - (1-x)\exp(\frac{-x}{\sqrt{\varepsilon}}) - x\exp(\frac{x-1}{\sqrt{\varepsilon}})$.

Table 1: Comparison of maximum absolute errors for Example 1

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256
Present Method					
2^{-7}	1.0094e-05	6.2910e-07	3.8387e-08	2.3714e-09	1.4743e-10
2^{-8}	4.3615e-05	2.7119e-06	1.6790e-07	1.0431e-08	6.5027e-10
2^{-9}	1.6512e-04	1.0743e-05	7.1676e-07	4.4779e-08	2.7952e-09
2^{-10}	5.4335e-04	4.6379e-05	2.9922e-06	1.8836e-07	1.1785e-08
Result in [12]					
2^{-7}	1.1522e-04	7.3531e-06	4.6790e-07	2.9236e-08	1.8268e-09
2^{-8}	5.6563e-04	4.1555e-05	2.7502e-06	1.7562e-07	1.1074e-08
2^{-9}	1.8378e-03	1.5766e-04	1.1083e-05	7.22s68e-07	4.5926e-08
2^{-10}	4.1400e-03	4.5262e-04	3.5704e-05	2.4223e-06	1.5601e-07

Table 2: Effects of using piecewise mesh of computed maximum absolute errors for Example 1

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512
Piecewise mesh (Present Method)					
10^{-2}	3.7103e-07	2.2627e-08	1.3959e-09	8.6527e-11	4.8122e-12
10^{-3}	4.4220e-05	2.8501e-06	1.7936e-07	1.1221e-08	7.0126e-10
10^{-4}	3.5718e-04	5.8120e-05	6.8177e-06	7.5244e-07	7.5760e-08
10^{-8}	3.6483e-04	6.0109e-05	7.0869e-06	7.8247e-07	7.8805e-08
10^{-12}	3.6491e-04	6.0129e-05	7.0896e-06	7.8277e-07	7.8836e-08
Uniform mesh (Present Method)					
10^{-2}	3.7103e-07	2.2627e-08	1.3959e-09	8.6527e-11	4.8122e-12
10^{-3}	4.4220e-05	2.8501e-06	1.7936e-07	1.1221e-08	7.0126e-10
10^{-4}	1.8681e-03	2.5441e-04	1.8402e-05	1.1965e-06	7.6331e-08
10^{-8}	2.6252e-02	2.6119e-02	2.5883e-02	2.5106e-02	2.2327e-02
10^{-12}	2.6268e-02	2.6184e-02	2.6140e-02	2.6117e-02	2.6106e-02

Table 3: Comparison of computed maximum absolute errors for Example 2

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512
Present Method					
2^{-12}	1.7032e-05	1.2354e-06	2.3333e-08	3.8167e-10	6.0567e-12
2^{-16}	1.6638e-05	1.4830e-06	6.8742e-08	2.5860e-09	8.3595e-11
2^{-20}	1.6540e-05	1.4715e-06	6.8214e-08	2.5652e-09	8.2911e-11
2^{-25}	1.6513e-05	1.4684e-06	6.8069e-08	2.5595e-09	8.2726e-11
Result in [11]					
2^{-12}	8.6985e-04	6.4050e-05	4.0017e-06	2.5010e-07	1.5632e-08
2^{-16}	8.1752e-04	8.4387e-05	7.8780e-06	6.8770e-07	5.7247e-08
2^{-20}	8.6985e-04	8.4387e-05	7.8536e-06	6.6828e-07	5.5642e-08
2^{-25}	8.6985e-04	8.4387e-05	7.8536e-06	6.6828e-07	5.5642e-08

Table 4: Computed maximum absolute errors and rate of convergence for both Example 1 and 2

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512
Example 1					
10^{-2}	3.7103e-07	2.2627e-08	1.3959e-09	8.6527e-11	4.8122e-12
	4.0354	4.0188	4.0119	4.0004	
10^{-3}	4.4220e-05	2.8501e-06	1.7936e-07	1.1221e-08	7.0126e-10
	3.1951	3.9901	3.9986	4.0001	
10^{-4}	3.5718e-04	5.8120e-05	6.8177e-06	7.5244e-07	7.5760e-08
	2.6195	3.0917	3.1796	3.3121	
Example 2					
10^{-2}	6.0599e-07	3.8082e-08	2.3834e-09	1.4905e-10	9.2980e-12
	3.9921	3.9980	3.9992	4.0027	
10^{-3}	5.1858e-05	3.3782e-06	2.1378e-07	1.3405e-08	8.3857e-10
	3.9402	3.9821	3.9953	3.9987	
10^{-4}	3.6928e-04	6.1080e-05	7.1984e-06	7.9560e-07	8.0158e-08
	2.5959	3.0850	3.1776	3.3111	

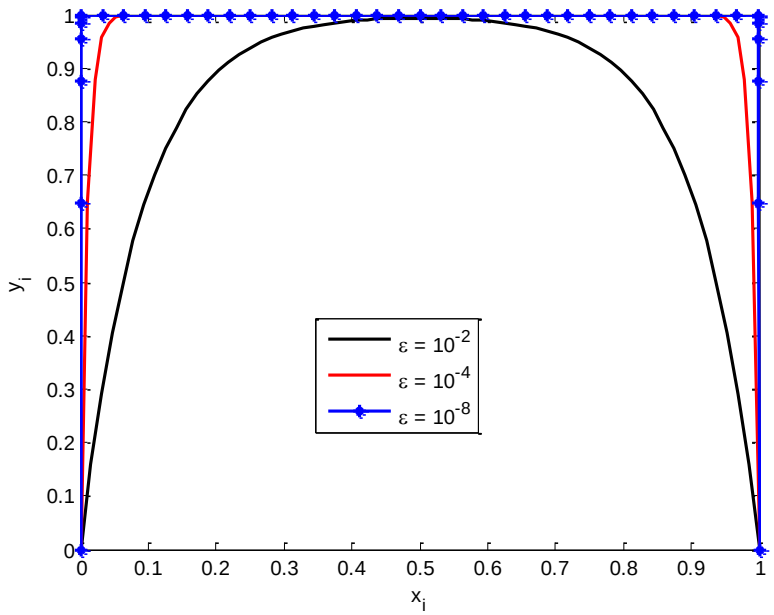


Figure 1: Effects of perturbation parameter on the behavior of the solution for Example 1, when $N = 64$.

4.8 **Discussion**

This study introduces a computational method based on a piecewise mesh to solve singularly perturbed reaction-diffusion problems. The solution domain is first discretized according to the lengths of the piecewise mesh, and the derivatives are replaced with finite difference approximations derived from Taylor series expansions. As demonstrated in Eq. (4.13), this approach results in a system of linear algebraic equations. Due to the diagonal dominance of the coefficient matrix, where the sum of the off-diagonal elements is less than or equal to the diagonal element, the system can be effectively solved using the Thomas algorithm. The matrix form presented in Eq. (4.24) confirms the stability of the proposed scheme.

To validate the method, several modeled examples are provided, with numerical results presented in Tables 1–3 and Figure 1. Figure 1 particularly highlights the effects of the perturbation parameter and the behavior of boundary layers. However, as shown in Table 2, when using a uniform mesh, the method struggles to produce accurate results in the presence of a small positive perturbation parameter. To address this, a piecewise mesh with finer resolution in the boundary layer region and coarser resolution in the outer layer is used.

The maximum absolute errors and convergence rates, displayed in Table 4, support the theoretical claims of the scheme. Additionally, Tables 1 and 3 show that the proposed method provides more accurate solutions compared to existing methods in the literature. The results also indicate that as the number of mesh point’s increases, the maximum absolute error decreases, demonstrating the convergence and effectiveness of the method.

Chapter Five

Conclusion and Future Work

5.1. Conclusion

A computational finite difference approach, using a piecewise mesh, has been developed to solve singularly perturbed reaction-diffusion problems. The accuracy of the solution is further enhanced by applying Richardson extrapolation techniques. Several numerical examples have been modeled to demonstrate the method's effectiveness. The results show that the method achieves lower maximum absolute errors and a higher order of convergence, confirming its efficiency.

Additionally, the proposed method provides more accurate solutions than some existing methods in the literature. Numerical experiments verify that it achieves better convergence with fewer errors, aligning with the theoretical predictions. Thus, the method offers an effective approximation of the exact solution.

5.2 Future Work

This study presents a computational approach on a piecewise mesh for solving singularly perturbed reaction-diffusion problems. Future research could focus on extending the proposed approach to higher-order convergence methods to achieve even more precise solutions for similar problems.

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