

**Numerical Method For Solving Singularly Perturbed two parameter Delay  
Differential Equation using Non-standard Finite Difference Method**



**College of Natural Science  
Department of Mathematics**

**A Thesis Submitted of the Department of Mathematics Jimma University  
in Partial Fulfillment of the Requirements of Degree of Master of Science in  
Mathematics  
(Numerical Analysis)**

**By: Yeshimebet Shiferaw Legesse  
Advisor: HabtamuGaroma(Ph.D)  
Co-Adivisor :Masho Jima(Ph.D)**

**May, 2024  
Jimma, Ethiopia**

# DECLARATION

I undersigned declare that, this research proposal entitled “Numerical Method For Solving Singularly Perturbed two parameter Delay Differential Equation using Non-standard Finite Difference Method” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Name: Yeshiemebet Shiferaw

Signature:

Date:

# ADVISORS'THESIS APPROVAL SHEET

This is to certify that the thesis entitled "Numerical Method For Solving Singularly Perturbed two parameter Delay Differential Equation using Non-standard Finite Difference Method has been developed by **Yeshiemebet Shiferaw**, under my supervision. Therefore We recommend that the student's proposal can be presented for review and open oral presentation.

**Habtamu Garoma(PhD)**

|                           |           |      |
|---------------------------|-----------|------|
| Name of Principal Advisor | Signature | Date |
|---------------------------|-----------|------|

**Masho Jima(PhD)**

|                    |           |      |
|--------------------|-----------|------|
| Name of Co-advisor | Signature | Date |
|--------------------|-----------|------|

## Acknowledgment

First of all, I am indebted to my almighty God. For giving long live, strength and helped me to reach this precise time. Next, I would like to thank, my respectable Advisor Dr. Habtamu Garoma and my co-advisor Dr. Masho Jima. I would like to thank my all friends in moral support and giving strength during the study period. Take this opportunity to express my gratitude to mathematics department of Jimma University.

# Contents

|  |           |
|--|-----------|
| Declaration . . . . .  | i         |
| Approval Sheet . . . . .                                       | ii        |
| Acknowledgment . . . . .                                       | iii       |
| Table of Content . . . . .                                     | v         |
| List of tables . . . . .                                       | vi        |
| List of Figures . . . . .                                      | vii       |
| Abbreviations and Acronyms . . . . .                           | viii      |
| Abstract . . . . .   | ix        |
| <b>1 INTRODUCTION</b>  | <b>1</b>  |
| 1.1 Background of the Study . . . . .                          | 1         |
| 1.2 Statement of the Problem . . . . .                         | 3         |
| 1.3 Objectives of the study . . . . .                          | 4         |
| 1.3.1 General Objective . . . . .                              | 4         |
| 1.3.2 Specific Objectives . . . . .                            | 4         |
| 1.4 Significance of the Study . . . . .                        | 5         |
| 1.5 Delimitation of the study . . . . .                        | 5         |
| <b>2 LITERATURE REVIEW</b>                                     | <b>7</b>  |
| 2.1 Singularly Perturbed Delay Differential Equation . . . . . | 7         |
| 2.2 Boundary Value Problem . . . . .                           | 8         |
| 2.3 Recent Development . . . . .                               | 9         |
| <b>3 MATERIALS AND METHODS</b>                                 | <b>10</b> |
| 3.1 Study Area and Period . . . . .                            | 10        |

|          |   |           |
|----------|---|-----------|
| 3.2      | Study Design . . . . .                                    | 10        |
| 3.3      | Sources of data . . . . .                                 | 10        |
| 3.4      | Mathematical procedure . . . . .                          | 10        |
| <b>4</b> | <b>DESCRIPTION OF THE METHODS, RESULTS AND DISCUSSION</b> | <b>12</b> |
| 4.1      | Description of the method . . . . .                       | 12        |
| 4.1.1    | Properties of Continuous solution . . . . .               | 13        |
| 4.2      | Formulation of the method . . . . .                       | 16        |
| 4.3      | Uniform Convergence Analysis . . . . .                    | 19        |
| 4.4      | Numerical Examples and Results . . . . .                  | 23        |
| <b>5</b> | <b>Conclusion and scope for Future work</b>               | <b>25</b> |
| 5.1      | Conclusion . . . . .                                      | 25        |
| 5.2      | SCOPE FOR FUTURE . . . . .                                | 25        |
|          | <b>References</b>   | <b>25</b> |

# List of Tables

|     |   |    |
|-----|---|----|
| 4.1 | Maximum absolute error and rate of convergence for different values of $N$ and $\varepsilon$ with non-standard fitted operator method for the give Example 4.1  | 24 |
| 4.2 | Comparison of Maximum absolute error and rate of convergence for different values of $\mu$ and $N$ the with non-standard fitted operator method and fitted mesh method for the give Example 4.1 . . . . . | 24 |

# List of Figures

|     |   |    |
|-----|---|----|
| 5.1 | The behavior of the Numerical Solution for Example 4.1 at $\varepsilon = 2^{-34} \rightarrow \varepsilon = 2^{-43}$ $N$ and $2N$ . . . . .                      | 26 |
| 5.2 | The behavior of the Numerical Solution for Example 4.1 at $\varepsilon = 2^{-32}$ and $N = 32$ . . . . .  | 26 |
| 5.3 | $\varepsilon$ -uniform convergece with fitted operator in log-log scale for Example 4.5.1 at $\varepsilon=2^{-34}$ $N = 64$ , $N = 128$ and $N = 256$ . . . . . | 27 |

# Abbreviations and Acronyms

**SPDDE** - Singularly Perturbed Delay Differential Equation

**BVP** - Boundary Value Problem

**DE** - Differential Equation

**DDEs** - Delay Differential Equations

**SP** - Singularly Perturbed

**SPPs** - Singularly Perturbed Problems

**ODE** - Ordinary Differential Equation

**PDE** - Partial Differential Equation

**NDDE** - Neutral Delay Differential Equation

**RDDE** - Retarded Delay Differential Equation

**SCD** - Stable Central Difference Scheme.

## Abstract

*Due to the wider applications of singularly perturbed differential equation (SPDDEs) in the real life problems, there has been growing interest to develop numerical methods for SPDDEs. However, as the perturbation parameter decreases in magnitude, standard numerical methods for solving singularly perturbed differential equations are unstable, inefficient and inaccurate. Thus, it is pretty reasonable to construct numerical methods whose convergence properties, accuracy and the computation cost are independent of the value of the perturbation parameter. One of the approaches to derive  $\epsilon$ -uniform convergent numerical methods for solving SPDDEs is through the application of fitted operator method which can be constructed by using either exponentially fitted operator or non-standard finite difference methods. The design of the non-standard finite difference methods (NSFDMs) starts mostly with the concept of exact schemes. A major advantage of having an exact difference scheme for a differential equation is that questions related to the usual considerations of consistency, stability and convergence do not arise. The objective of this project is, therefore, to construct  $\epsilon$ -uniform convergent non-standard finite difference method for solving singularly perturbed two parameter delay differential equation using finite difference method. The convergence analysis and stability of the proposed method will also be established.*

# Chapter 1

## INTRODUCTION

### 1.1 Background of the Study

Numerical analysis is concerned with mathematical derivation of numerical methods, designing an algorithm for the method, implementation of these algorithms on computer and analyses, of the error associated with the method, all to solve mathematical problem. An equation involving one independent variable and its derivative with respect to one or more independent variable is called differential equation. There are two main class of differential equation. these are ordinary differential equation and partial differential equation.

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation (ODE). In general, the differential equations in which the highest order derivative term is multiplied by a small positive parameter  $\varepsilon$  where  $0 < \varepsilon \ll 1$  are known to be singularly perturbed differential equations, and the parameter is known as the perturbation parameter. The classification of singularly perturbed higher order problems depending on how the order of the original equation is affected if one sets  $\varepsilon = 0$ , where  $\varepsilon$  is small positive parameter multiplying the highest derivative occurring in the differential equation. If the order is reduced by one, we say that the problem is convection-diffusion type and if the order is reduced by two, we say the problem is reaction-diffusion type. Any system involving a feedback control will almost involve time delays.

These arise because a finite time is required to sense information and then react to it. If we restrict the class of delay differential equation to a class in which the highest derivative is multiplied by a small positive parameter and involving at least one delay term, then it is said to be singularly perturbed delay differential equation. We call delay differential equations retarded type if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral delay differential equations. As  $\varepsilon$  is equal to zero, the solution of problems exhibits interesting behaviors (rapid changes) since the order of the equation reduces. The region where these rapid changes occur is called inner region and the region in which the solution changes regularly is called outer region. Singular perturbation problems arises very frequently in diversified field of applied mathematics and engineering. Such problems originate generally from real-world applications of algebra, geometry and calculus, and they involve variables which vary continuously; these problems occur throughout the natural sciences, social sciences, engineering, medicine. More detail about these problem can be found in Roos *et al.* (2018) and kumar *et al.*(2007). Due to the wider applications of SPDDEs in the real world problems, there has been growing interest to develop numerical methods for SPDDE. Cimen (2018) used a difference method on a piece-wise uniform mesh of Shishkin type for solving the singularly perturbed boundary value problem for a non-linear second-order delay differential equation. In Rai and Sharma (2020), the authors applied a standard finite difference scheme on a piece-wise uniform Shishkin mesh fitted to the boundary and interior layer regions combined with interpolation to overcome the effect of the delay term. In Kadelbajoo and Ramesh (2007), the authors studied a two-point boundary value problem for a second-order singularly perturbed differential equation with delay in both the convection and reaction terms using a Shishkin mesh with a grid adaptation strategy. Liu and Chen (2014) also applied an adaptive grid method for the numerical study of a singularly perturbed differential equation with small delay. Geng and Qian (2015) used a modified method for the numerical treatment of singularly perturbed boundary value problems with a delay. Patidar and Sharma (2006) constructed a  $\varepsilon$ -uniform convergent non-standard finite difference method for singularly perturbed differential equations with delay and advance arguments. Duressa and Debela (2021) introduced a non-standard fitted operator finite difference method for solving singularly perturbed differential equa-

tions involving large and small parameters.

Most of the classical numerical methods are not suitable for solving singularly perturbed problems (SPPs) due to the presence of the perturbation parameter  $\varepsilon$ , which leads to divergent approximate solutions Kumar and Kadalbajoo(2011). One natural way to overcome these divergences is to use a large number of mesh points when  $\varepsilon$  is very small. This is difficult and sometimes impossible to handle such cases. Therefore, it is necessary to develop suitable numerical methods which are uniformly convergent to solve this type of differential equations.

## 1.2 Statement of the Problem

The numerical treatment of a parameter uniform numerical method for singularly perturbed two parameter delay differential equation problems yields, the full equation including the function involved the value of the parameters of initial condition specific any boundary condition. The field of delay differential equation (DDE) attracted mathematicians and Engineers due to the following reasons. Firstly, we have to find an appropriate approximation of the solution at the delayed arguments. Secondly, the algorithm has to take care of the jump in the discontinuity due to the delay parameter and thirdly, its solution behavior is very interesting with boundary layers, interior layers and oscillations. As decreases in magnitude, standard numerical methods for solving singularly perturbed differential equations are unstable, inefficient and inaccurate Habtamu and Gamachis (2019), Gamechs and Reddy (2013). Thus, it is reasonable to construct numerical methods whose convergence properties, accuracy and the computation cost are independent of the value of the perturbation parameter. One of the approaches to derive -uniform convergent numerical methods is through the application of fitted operator difference method Debela and Duressa (2019) and Ranjan and Prasad(2014) which can be constructed by using either exponentially fitted operator or non-standard difference methods Patidar (2005). In Cimen (2018),Cakr (2016),Cimen(2017),Selvi and Ramanujan(2016) different techniques to the design and analysis of suitable numerical methods for singularly perturbed differential equations involving delay parameter have been considered. A uniform parameter convergent and stable accelerated fitted operator method

is, proposed in Miller (2012). For singularly perturbed delay differential equations, with non-local boundary conditions. There are two principal approaches to deal with singularly perturbed problems, namely: the fitted operator methods and the fitted mesh methods . In order to design appropriate methods, which belong to the latter class, one must have some a priori information about the sensitive regions where the exact solution undergoes some rapid changes. Very often, it is not possible to have such a priori information and that is why the fitted operator type of methods is useful. According to Patidar in (2005), the design of the non-standard finite difference methods (NSFDMs) starts mostly with the concept of exact schemes. An exact difference scheme for a differential equation delivers a perfect match between the discrete and continuous solutions. The goal of this project is, to construct  $\epsilon$ -uniform convergent non-standard finite difference method for solving singularly perturbed delay differential equation, using non-standered finite difference method. The convergence analysis and stability of the proposed method will also be established. In order to achieve the objective, the study will attempts to answer the following questions:

- How to construct a parameter uniform numerical method for singularly perturbed two parameter delay deferential equation using non-standered finite difference method.
- How can we investigate the convergent scheme .
- How can we establish the approximates of the solutions

## 1.3 Objectives of the study

### 1.3.1 General Objective

The general objective of this study is to develop a parameter uniform method for singularly perturbed two parameter delay differential equation using non-standered finite difference method.

### 1.3.2 Specific Objectives

The study has the following specific objectives:

1. To construct a parameter uniform numerical method for singularly perturbed two parameter delay differential equation using non-standard finite difference method.
2. To investigate the convergent scheme .
3. To establish the accuracy of the solutions.

## 1.4 Significance of the Study

The significance of the study can be listed as follows:

- Developing a numerical method for solving singularly perturbed two-parameter delay differential equations is a significant contribution to the field of applied numerical mathematics.
- By studying the behavior of singularly perturbed delay differential equations, researchers can gain insights into the dynamics of complex systems with small perturbation parameters.
- The developed numerical method has practical applications in solving real-world problems involving singularly perturbed delay differential equations.
- The insights and techniques developed in this study can be generalized to other singularly perturbed systems beyond the specific problem addressed.
- To help the graduate student to acquire research skills and scientific procedures.

## 1.5 Delimitation of the study

Consider the singularly perturbed two-parameter large delay differential equation.

$$Lu \equiv \begin{cases} \varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) + c(x)u(x-1) = f(x), & x \in \Omega, \\ u(x) = \phi(x) \text{ on } [-1, 0], \\ u(2) = l, \end{cases}$$

where  $\Omega = (0, 2)$  and  $0 < \varepsilon \ll 1$ ,  $0 < \mu < 1$  are two small parameters.  $\Omega_1 = (0, 1]$ ,  $\Omega_2 = (1, 2)$ ,  $\Omega_0 = [-1, 0]$  and  $l$  is a given constant.

The coefficient functions  $a(x)$ ,  $b(x)$ ,  $c(x)$ , and  $f(x)$  are sufficiently smooth on  $\bar{\Omega} = [0, 1]$ , and  $a(x) \geq \alpha > 0$ ,  $b(x) \geq \beta > 0$ ,  $c(x) > 0$ , and  $(b - c)(x) \geq k > 0$ .  $\phi(x)$  is sufficiently smooth on  $[-1, 0]$ . The solution of this problem exhibits boundary layers at  $x = 0$  and  $x = 2$ , and interior layers at  $x = 1$ .

# Chapter 2

## LITERATURE REVIEW

### 2.1 Singularly Perturbed Delay Differential Equation

An equation in which evolution of system at a certain time depends on the rate at an earlier time is called Singularly perturbed delay differential equation Miller *et al.*( 2009). The delay differential equation in which the highest derivative is multiplied by perturbation parameter is known as perturbed delay differential equation. The delay differential equation can be classified as retarded delay differential equation and neutral differential equation. A delay differential equation is said to be of retarded delay differential equation (RDDE) if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral delay differential equations (NDDE). If we restrict it to a class in which the highest derivative term is multiplied by a small parameter, then we obtain singularly perturbed delay differential equation of the retarded type.

Frequently, delay differential equations have been reduced to differential equations with coefficients that depend on the delay by means Taylor's series expansions of the terms that involve delay and the resulting differential equation have been solved either analytically when the coefficients of these equations are constant or numerically, when they are not. The theory and numerical solution of singularly perturbed delay differential equation are still at the initial stage. A fitted-stable central difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) of the interval.

## 2.2 Boundary Value Problem

A boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem Keller (1992).

A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions. A boundary condition is a prescription some combinations of values of the unknown solution and its derivatives at more than one point. Finding the numerical solution of a boundary value problem is more difficult than that of corresponding initial value problem. For, there are BVPs for which solutions do not exist; and even if a solution exists there might be many more. Thus, existence and uniqueness generally fail for BVPs. Finding the solution of singularly perturbed delay differential equations is a challenging problem. In response to these, in recent years there has been a growing interest in numerical methods on singularly perturbed delay differential equations. In Kadalbajoo and Sharma (2002) initiated the numerical study of such type of boundary value problems. Liu and Chen(2014), Patidar and Sharma (2006), In a fitted operator scheme on a uniform mesh is suggested to solve an initial value problem for a class of linear first order delay differential equations. Amiraliyev and Cimen (2018) proposed a first order uniform convergent fitted finite difference scheme for singularly perturbed boundary value problem for a linear second order delay differential equation with large delay in reaction term. Patidar and Sharma in (2006) combined, fitted operator methods with, Mickens nonstandard finite difference techniques for the numerical solution of singularly perturbed linear ordinary differential equations by means of first-order Taylor's series expansion of the terms involving shifts that result in linear ordinary differential equations. Mohapatra and Natesan (2011) proposed the upwind finite difference operator on an adaptive and grid for a class of singularly perturbed differential equations with small delay and shift terms. Cakir (2016) studied the numerical solution of singularly perturbed problem with integral boundary condition finite difference method on a piece wise uniform mesh of Shishkin type. The numerical and analytical treatments of singularly perturbed problems are far from triv-

ial due to the boundary layer. behavior of the solution Rajan and Prasad(2021). The numerical approximation of DDEs is challenging to apply an appropriate method for the retarded arguments and the algorithm needs to take into account the jump discontinuity due to the delay term Rai and Sharma (2012). The major drawback of Shishkin meshes is the requirements of prior information of the width/thickness and location of the layer regions Podila and Kumar (2019).

## 2.3 Recent Development

Langa and Miura (1994) gave an asymptotic approximation to solve singularly perturbed second-order delay differential equations. Chakravarthy *et al.* (2015), dealt with the singularly perturbed boundary value problem for the second-order delay differential equation. Similar boundary value problems are associated with expected first-exit times of the membrane potential in models of neurons. Chakravarthy *et al.* (2017) dealt with a singularly perturbed boundary value problem for a linear second-order delay differential equation.

Kumar and Rao (2020), presented a stabilized central difference method for the boundary value problem of singularly perturbed differential equations with a large negative shift. The central difference approximations for the derivatives are modified by re-approximating the error terms, leading to a stabilizing effect. The method is found to be second-order convergent. As introduced in the literature, most researchers have tried to find approximate solutions for singularly perturbed differential equations with a large delay, but mainly focusing on constant coefficients, and some others who have worked on variable coefficients did not obtain more accurate solutions. Owing to this, this study presents a more accurate and convergent numerical method for singularly perturbed differential equations with two parameters and large delay, by using a nonstandard finite difference method.

# Chapter 3

## MATERIALS AND METHODS

### 3.1 Study Area and Period

The study was conducted at Jimma University under the College of Natural sciences in Mathematics department from January 2024 to June 2024 G.C. conceptually the thesis will focus on solving numerical method for singularly perturbed two parameter delay differential equation using non-standard finite difference method .

### 3.2 Study Design

This study will employ mixed design review of the existing literature, and execution and simulation of numerical examples for validating the proposed method.

### 3.3 Sources of data

The relevant source of data for this study are books, published articles on reputable journals and related study from Internet.

### 3.4 Mathematical procedure

To achieve the stated objectives, the study was followed the following steps:

1. Defining (or describing) problem.

2. Discretizing the domain /interval.
3. Describing the method by non-standard finite difference method approximation and obtain the schemes.
4. Establishing the stability and convergence analysis of the formulated schemes.
5. Writing an algorithm for the developed schemes.
6. Writing MATLAB code for the numerical method.

# Chapter 4

## DESCRIPTION OF THE METHODS, RESULTS AND DISCUSSION

### 4.1 Description of the method

Consider singularly perturbed two-parameter large delay differential equation of the form.

$$\varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) + c(x)u(x-1) = f(x) \quad x \in \Omega, \quad (4.1)$$

subject to with the interval and boundary conditions

$$\begin{aligned} u(x) &= \phi(x) \quad \text{on } [-1, 0], \\ u(2) &= l, \end{aligned}$$

where  $\Omega = (0, 2)$  and  $0 < \varepsilon \ll 1$ ,  $0 < \mu < 1$  are two small parameters.  $\Omega_1 = (0, 1)$ ,  $\Omega_2 = (1, 2)$ ,  $\Omega_0 = [-1, 0]$  and  $l$  is a given constant.

The above equation is written as in the form of

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) + c(x)u(x-1) = f(x) \quad x \in \Omega, \quad (4.2)$$

where  $p(x) = \mu a(x)$  and  $q(x) = -b(x)$

From Eq.(4.2) the values of  $u(x-1)$  is known for the domain  $\Omega_1 = (0, 1]$  and unknown

for the  $\Omega_2 = (1, 2)$  due to the large delay at  $x = 1$ .

Thus we have to treat the problem at  $\Omega_1$  and  $\Omega_2$  separately :

we can write Eq(4.2)in operator form as:

$$Lu(x) = R(x) \quad (4.3)$$

where

$$Lu(x) = \begin{cases} L_1u(x) = \varepsilon u''(x) + p(x)u'(x) + q(x)u(x); & x \in \Omega_1, \\ L_2u(x) = \varepsilon u''(x) + p(x)u'(x) + q(x)u(x) + c(x)u(x-1), & x \in \Omega_2, \end{cases} \quad (4.4)$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1); & x \in \Omega_1, \\ f(x), & x \in \Omega_2, \end{cases} \quad (4.5)$$

with interval and boundary conditions:

$$\begin{cases} u(x) = \phi(x); & x \in [-1, 0], \\ u(1^-) = u(1^+); & u'(1^-) = u'(1^+), \\ u(2) = l. \end{cases} \quad (4.6)$$

where  $u(1^-)$  and  $u(1^+)$  denoted the left and right limits of  $u(x)$  at  $x = 1$ , respectively.

#### 4.1.1 Properties of Continuous solution

**Lemma 1 (Maximum Principle):** Let  $\psi(x)$  be any function in  $X$  such that  $\psi(0) \geq 0$ ,  $\psi(2) \geq 0$ ,  $L_1\psi(x) \geq 0 \forall x \in \Omega_1$ ,  $L_2\psi(x) \geq 0 \forall x \in \Omega_2$ , and  $\psi'(1^+) - \psi'(1^-) = \psi'(1) \leq 0$ . Then  $\psi(x) \geq 0$  for all  $x \in \bar{\Omega}$ .

**Proof:** We define the test function  $s(x)$  as follows:

$$s(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, & \text{for } x \in [0, 1] \\ \frac{3}{8} + \frac{x}{4}, & \text{for } x \in [1, 2] \end{cases} \quad (4.7)$$

Note that  $s(x) > 0, \forall x \in \bar{\Omega}$ ;  $Ls(x) > 0, \forall x \in \Omega_1 \cup \Omega_2, s(0) > 0, s(2) > 0,$  and  $s'(1) < 0$ .

Let  $\mu = \max\{\frac{-\psi(x)}{s(x)} : x \in \bar{\Omega}\}$ . Then, there exists  $x_0 \in \bar{\Omega}$  such that  $\psi(x_0) + \mu s(x_0) = 0$  and  $\psi(x) + \mu s(x) \geq 0, \forall x \in \bar{\Omega}$ .

Therefore, the function  $(\psi + \mu s)$  attains its minimum at  $x = x_0$ .

Suppose the theorem does not hold true, then  $\mu > 0$ .

**Case 1:**  $x_0 = 0$

$$0 < (\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0.$$

It is a contradiction.

**Case 2:**  $x_0 \in \Omega_1$

$$0 < L(\psi + \mu s)(x_0) = \varepsilon(\psi + \mu s)''(x_0) + p(x_0)(\psi + \mu s)'(x_0) + q(x_0)(\psi + \mu s)(x_0) \leq 0.$$

It is a contradiction.

**Case 3:**  $x_0 = 1$

$$0 \leq (\psi + \mu s)'(1) = \psi'(1) + \mu s'(1) < 0.$$

It is a contradiction.

**Case 4:**  $x_0 \in \Omega_2$

$$0 < L(\psi + \mu s)(x_0) = \varepsilon(\psi + \mu s)''(x_0) + p(x_0)(\psi + \mu s)'(x_0) + q(x_0)(\psi + \mu s)(x_0) + c(x_0)(\psi + \mu s)(x_0 - 1) \leq 0.$$

It is a contradiction.

**Case 5:**  $x_0 = 2$

$$0 < (\psi + \mu s)(2) = (\psi + \mu s)(2) \leq 0.$$

It is a contradiction.

Hence, the proof of the theorem.

**Lemma 2 (Stability Result):** The solution  $u(x)$  of the problem (4.1) satisfies the bound

$$|u(x)| \leq C \max \left\{ |u(0)|, |u(2)|, \sup_{x \in \Omega_*} |u(x)| \right\}, \quad x \in \bar{\Omega}.$$

This theorem can be proved by using Lemma 1 and the barrier functions  $\theta^\pm = CMs(x) \pm u(x)$ ,  $x \in \bar{\Omega}$ ,

where  $M = \max \{ |u(0)|, |Lu(2)|, \sup_{x \in \Omega_*} |Lu(x)| \}$

and

$s(x)$  is the test function as in Lemma 1.

The uniqueness of the solution is implied by this minimum principle. Its existence follows trivially (as for linear problems, the uniqueness of the solution implies its existence). This principle is now applied to prove that the solution of Eq. (4.1) is bounded. The following lemma shows the bound for the derivatives of the solution.

**Lemma 3:** Let  $u(x)$  be the solution of Eq. (4.1) Then, we have the following bounds:

$$|u^{(k)}(x)|_{\Omega_*} \leq C\varepsilon^{-k}, \quad \text{for } k = 1, 2, 3.$$

**Proof:** To bound  $u'(x)$  on the interval  $\Omega_1 = (0, 1)$ , we consider

$$L_1 u(x) \equiv \varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = f(x) - c(x)\phi(x-1).$$

Integrating the above equation on both sides on we have

$$\varepsilon [u'(x) - u'(0)] = - [p(x)u(x) - p(0)u(0)] + \int_0^x p'(t)u(t)dt - \int_0^x q(t)u(t)dt + \int_0^x [f(t) - c(t)\phi(t-1)]dt,$$

Therefore,

$$\varepsilon u'(0) = \varepsilon u'(x) - [p(x)u(x) - p(0)u(0)] + \int_0^x p'(t)u(t)dt - \int_0^x q(t)u(t)dt + \int_0^x [f(t) - c(t)\phi(t-1)]dt,$$

Then by the Mean Value Theorem, there exists  $z \in (0, \varepsilon)$  such that

$$|\varepsilon u'(z)| \leq C (|u(x)|, |R(x)|, |\phi(x)|_{[-1,0]}),$$

and  $|\varepsilon u'(0)| \leq C(|u(x)| + |R(x)| + |\phi(x)|)$ .

Hence,

$$|u'(x)| \leq C \max(|u(x)|, |R(x)|, |\phi(x)|).$$

By a similar argument, we can bound  $u'(x)$  on  $(1, 2)$  as  $|\varepsilon u'(x)| \leq C$ .

From Eq. (4.3) and (4.4), we have

$$|u^{(k)}(x)|_{\Omega^*} \leq C\varepsilon^{-k} \text{ for } k = 2, 3, 4.$$

Hence, the proof.

## 4.2 Formulation of the method

The theoretical basis of non-standard discrete numerical method is based on the development of exact finite difference method. Mickens (1991) presented techniques and rules for developing non-standard finite difference methods for different problem types. In Mickens's rules, to develop a discrete scheme, denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard procedures. These complicated functions constitute a general property of the schemes, which is useful while designing reliable schemes for such problems. For the problem of the form in Eq. (4.1), in order to construct exact finite difference scheme, we follow the procedures used in Bansal and Sharma (2017). Let us consider the following singularly perturbed differential equation of the form

$$\begin{cases} \varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = R(x), & x \in (0, 1); \\ u(0) = \phi(0), \quad u(1) = \gamma; \end{cases} \quad (4.8)$$

where  $R(x) = f(x) - c(x)\phi(x - 1)$ .

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = 0 \quad (4.9)$$

$$\varepsilon u''(x) + p(x)u'(x) = 0 \quad (4.10)$$

The constant coefficient homogeneous problems corresponding to Eq. (4.9) are:

where  $\lambda = \frac{q \pm \sqrt{q^2 - 4p\varepsilon}}{2\varepsilon}$ . Two linear independent solutions of Eq. (4.10) are  $\exp(\lambda_1 x)$  and  $\exp(\lambda_2 x)$ , where

$$\lambda_{1,2} = \frac{-q \pm \sqrt{q^2 - 4p\varepsilon}}{2\varepsilon}. \quad (4.11)$$

We discretize the domain  $[0, 1]$  using uniform mesh length  $\Delta x = h$  such that

$$p(x)u'(x) + q(x)u(x) = R(x), \quad u_0 = \phi(0). \quad (4.12)$$

We discretize the domain  $[0, 1]$  using a uniform mesh length  $\Delta x = h$  such that  $\Omega_1^N = \{x_i = x_0 + ih | i = 0, 1, 2, \dots, N\}$ , where  $x_0 = 0$ ,  $x_N = 1$ , and  $h = \frac{1}{N}$ , and  $N$  denotes the number of mesh points.

We denote the approximate solution to  $u(x)$  at grid point  $x_i$  by  $U_i$ .

Now our objective is to calculate a difference equation which has the same general solution as Eq. (4.7) at the grid point  $x_i$  given by:

$$U_i = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i) \quad (4.13)$$

Using the procedures used in , we have: By considering a small enough  $h$ , the discretized form of Eq. (11) becomes

$$\det \begin{pmatrix} U_{i-1} & \exp(\lambda_1(x_{i-1})) & \exp(\lambda_2(x_{i-1})) \\ U_i & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ U_{i+1} & \exp(\lambda_1(x_{i+1})) & \exp(\lambda_2(x_{i+1})) \end{pmatrix} = 0 \quad (4.14)$$

Simplifying Eq. (13), we obtain

$$-\exp\left(-\frac{ph}{2\varepsilon}\right) U_{i-1} + 2 \cosh\left(\frac{h\sqrt{q^2 - 4p\varepsilon}}{2\varepsilon}\right) U_i - \exp\left(\frac{-ph}{2\varepsilon}\right) U_{i+1} = 0 \quad (4.15)$$

which is an exact difference scheme for Eq.(14) Since  $\varepsilon \rightarrow 0$ , we use an approximation  $\frac{h\sqrt{q^2 - 4p\varepsilon}}{2\varepsilon} \approx$

$\frac{ph}{2\varepsilon}$  in Eq.(14) Hence, multiplying both  $\exp\left(\frac{ph}{2\varepsilon}\right)$

$$\varepsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{\frac{h\varepsilon}{p} \left( \exp\left(\frac{ph}{\varepsilon}\right) - 1 \right)} + p \frac{U_{i+1} - U_i}{h} = 0 \quad (4.16)$$

$$(4.17)$$

The denominator function becomes  $\psi^2 = \frac{h\varepsilon}{p} \left( \exp\left(\frac{ph}{\varepsilon}\right) - 1 \right)$ . Adopting this denominator function for the variable coefficient problem, we write it as

$$\psi_i^2 = \frac{h\varepsilon}{p_i} \left( \exp\left(\frac{p_i h}{\varepsilon}\right) - 1 \right) \quad (4.18)$$

where  $\psi_i^2$  is a function of  $\varepsilon, p_i$  and  $h$  where  $p_i$  is denoted for  $p(x_i)$

Assume that  $\overline{\Omega}^{2N}$  denote the partition of  $[0, 2]$  into  $2N$  subintervals such that:

$$0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1 \quad 1 < x_{N+1} < x_{N+2} < \dots < x_{2N-1} < x_{2N} = 2$$

where  $x_i = ih$  and  $h = \frac{2}{2N} = \frac{1}{N}$  for  $i = 0, 1, 2, \dots, 2N$ .

**Case (1):** Consider Eq(4.8). on the domain  $\Omega_1 = (0, 1]$ , which is given by

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = f(x) - c(x)\phi(x-1) \quad (4.19)$$

Undertaking the notation  $u_i \simeq u(x_i)$  and using the nonstandard finite difference methodology of Mickens (1994), for right layer in the domain  $\Omega_1$  the scheme to solve Equation (4.16) is given by

$$\varepsilon \left( \frac{U_{i-1} - 2U_i + U_{i+1}}{\psi_i^2} \right) + p_i \frac{U_{i+1} - U_i}{h} + q_i u_i \tau_1 = f_i - c_i \phi(x-1) \quad (4.20)$$

where

$$\psi_i^2 = \frac{h\varepsilon}{p_i} \left( \exp\left(\frac{p_i h}{\varepsilon}\right) - 1 \right) = h^2 + O\left(\frac{h^3}{\varepsilon}\right). \text{ the local truncation term } \tau_1 = \frac{hp_i}{2} u_i'' + o(h^2) = o(h)$$

The numerical scheme in Eq. (4.20) can be written in a three-term recurrence relation as

$$E_i U_{i-1} + F_i U_i + G_i U_{i+1} = H_i, \quad i = 1, 2, \dots, N-1 \quad (4.21)$$

where

$$E_i = \frac{\varepsilon}{\psi_i^2}, \quad F_i = -\frac{2\varepsilon}{\psi_i^2} - \frac{p_i}{h} + q_i, \quad G_i = \frac{\varepsilon}{\psi_i^2} + \frac{p_i}{h} \quad \text{and} \quad H_i = f_i$$

**Case (2):** Consider Eq.4.1 on the domain  $\Omega_2 = (1, 2)$ , for the right layer in the domain  $\Omega_2 = (1, 2)$  using the nonstandard finite difference method which is given by

$$\varepsilon \left( \frac{U_{i-1} - 2U_i + U_{i+1}}{\psi_i^2} \right) + p_i \frac{U_{i+1} - U_i}{h} + q_i u_i + c_i \phi(x-1) + \tau_1 = f_i \quad (4.22)$$

$$E_i U_{i-1} + F_i U_i + G_i U_{i+1} = H_i, \quad i = N+1, N+2, \dots, 2N-1 \quad (4.23)$$

where

$$E_i = \frac{\varepsilon}{\psi_i^2} - \frac{p_i}{h}, \quad F_i = \frac{2\varepsilon}{\psi_i^2} + \frac{p_i}{h} + q_i, \quad G_i = \frac{\varepsilon}{\psi_i^2} \quad \text{and} \quad H_i = f_i$$

Therefore, on the whole domain  $\bar{\Omega} = [0, 2]$ , the basic schemes to solve Eqs. (1)-(3) are the schemes given in Eqs. (21) and (23) together with the local truncation error of  $\tau_1$ .

### 4.3 Uniform Convergence Analysis

In this section, we need to show the discrete scheme in Eq. (??), satisfy the discrete minimum principle, uniform stability estimates, and uniform convergence.

**Lemma 4.4.1 (Discrete Minimum Principle)** Let  $U_i$  be any mesh function that satisfies:

$$\begin{aligned} U_0 &\geq 0, \\ U_N &\geq 0 \\ L_\varepsilon^N U_i &\leq 0, \quad \text{for } i = 1, 2, 3, \dots, N-1 \end{aligned}$$

then  $U_i \geq 0$ , for  $i = 1, 2, 3, \dots, N$ .

**Proof** is by contradiction. Let  $j$  be such that  $U_j = \min_i U_i$  and suppose that  $U_i \leq 0$ . Clearly,

$j \notin \{0, N\}$ . We have  $U_j - U_{j-1} \geq 0$  and  $U_{j+1} - U_j \leq 0$ . Therefore,

$$\begin{aligned} L_\varepsilon^N U_i &= \varepsilon \left( \frac{U_{j-1} - 2U_j + U_{j+1}}{\psi_i^2} \right) + p_j \frac{U_{j+1} - U_j}{h} \\ &= \frac{\varepsilon}{\psi_i^2} (U_{j-1} - 2U_j + U_{j+1}) + \frac{p_j}{h} (U_{j+1} - U_j) \\ &= \frac{\varepsilon}{\psi_i^2} ((U_{j+1} - U_j) - (U_j - U_{j-1})) + \frac{p_j}{h} (U_{j+1} - U_j) \geq 0. \end{aligned}$$

where the strict inequality holds if  $U_{j+1} - U_j > 0$ , this would lead to a contradiction, and therefore we can conclude that  $U_j \geq 0$ . Since  $j$  is arbitrary, we have  $U_i \geq 0$  for  $i = 0, 1, 2, \dots, N$ .

From the discrete minimum principle, we can obtain an  $\varepsilon$ -uniform stability property for the operator  $L_\varepsilon^N$ . The discrete operator  $L_\varepsilon^N$  is defined as follows:

The  $\varepsilon$  in the operator  $L_\varepsilon^N$  must satisfy the minimum principle in order to ensure the  $\varepsilon$ -uniform stability property. Next, we can analyze the uniform convergence of the operator  $L_\varepsilon^N$  as  $\varepsilon$  approaches 0 and  $N$  approaches infinity. This will provide insights into the properties and behavior of the discrete operator and its relationship to the continuous problem. Using Taylor series expansion, the bound for  $U(x_{i+1})$  and  $U(x_{i-1})$  at  $x_i$  as

$$\begin{cases} U(x_{i-1}) = U(x_i) - hU'(x_i) + \frac{h^2}{2!}hU''(x_i) - \frac{h^3}{3!}hU'''(x_i) + \frac{h^4}{4!}hU^{(4)}(x_i) + o(h^5) \\ U(x_{i+1}) = U(x_i) + hU'(x_i) + \frac{h^2}{2!}hU''(x_i) + \frac{h^3}{3!}hU'''(x_i) + \frac{h^4}{4!}hU^{(4)}(x_i) + o(h^5) \end{cases}$$

We obtain the bound for

$$\begin{cases} |D^+ D^- U(x_i)| \leq C|U''(x_i)| \\ |U''(x_i) - D^+ D^- U(x_i)| \leq Ch^2|U^{(4)}(x_i)| \end{cases} \quad (4.24)$$

Similarly, for the first derivative term

$$|U'(x_i) - D^+ U(x_i)| \leq Ch|U''(x_i)|$$

$$\text{where } |U^{(k)}(x_i)| = \sup_{x_i \in (x_0, x_N)} |U^{(k)}(x_i)|, \quad k = 2, 3, 4.$$

**Theorem 4.1** Let the coefficient functions  $a(x)$  and the source function  $f(x)$  in Eqs. (1.1)-(1.2) on the domain  $\Omega$  be sufficiently smooth, so that  $u(x) \in C^4[0, 1]$ . Then, the discrete solution  $U_i$  satisfies:

$$|L^N(u_i - U_i)| \leq Ch \left( 1 + \sup_{x \in (0,1)} \left( \frac{\exp(-\frac{px_i}{\varepsilon})}{\varepsilon^3} \right) \right)$$

We consider the truncation error discretization as

$$\begin{aligned} |L^N(u_i - U_i)| &= |L^N u_i - L^N U_i| \\ &\leq C \left| \varepsilon u_i'' + p_i u_i' - \left\{ \varepsilon \frac{D^+ D^- h^2}{\psi_i^2} u_i + p_i D^+ u_i \right\} \right| \\ &\leq C \left| \varepsilon \left( u_i'' + p_i u_i' - \frac{D^+ D^- h^2}{\psi_i^2} u_i \right) + p_i (u_i' - D^+ u_i) \right| \\ &\leq C\varepsilon |u_i'' - D^+ D^- u_i| + C\varepsilon \left| \left( \frac{h^2}{\psi_i^2} - 1 \right) D^+ D^- u_i \right| + Ch |u_i''| \\ &\leq C\varepsilon h^2 |u_i^{(4)}| + C\varepsilon |u_i''| + C\varepsilon |u_i''| \\ &\leq C\varepsilon h^2 |u_i^{(4)}| + C\varepsilon |u_i''| \end{aligned}$$

We used the estimate

$$\rho = \frac{p_i h}{\varepsilon}, \quad \rho \in (0, \infty).$$

$$\varepsilon \left| \frac{h^2}{\psi_i^2} - 1 \right| = \varepsilon \left| \frac{h^2}{\frac{h\varepsilon}{p_i} \left( \exp\left(\frac{p_i h}{\varepsilon}\right) - 1 \right)} - 1 \right| = p_i h \left| \frac{1}{\exp(\rho) - 1} - \frac{1}{\rho} \right| = p_i h Q(\rho)$$

By simplifying and writing explicitly we obtain

$$Q(\rho) = \frac{\exp(\rho) - \rho - 1}{\rho (\exp(\rho) - 1)}$$

and we obtain the limit is bounded as

$$\lim_{\rho \rightarrow 0} Q(\rho) = \frac{1}{2}, \quad \lim_{\rho \rightarrow \infty} Q(\rho) = 0$$

Hence,  $\forall \rho \in (0, \infty)$  we have  $Q(\rho) = C$  So, the error estimate in the discretization is bounded as

$$|L^N(u_i - U_i)| \leq C\varepsilon h^2 |u_i^{(4)}| + C\varepsilon |u_i''| \quad (4.25)$$

From Eq. (25) and boundedness of derivatives of solution in Lemma 3 , we obtain

$$\begin{aligned}
|L^N(u_i - U_i)| &\leq C\varepsilon h^2 \left| 1 - \varepsilon^{-4} \exp\left(\frac{p_i h}{\varepsilon}\right) \right| + Ch \left| 1 + \varepsilon^{-3} \exp\left(\frac{p_i h}{\varepsilon}\right) \right| \\
&\leq Ch^2 \left| 1 - \varepsilon^{-3} \exp\left(\frac{p_i h}{\varepsilon}\right) \right| + Ch \left| 1 + \varepsilon^{-2} \exp\left(\frac{p_i h}{\varepsilon}\right) \right| \\
&\leq Ch \left( 1 + \sup_{x \in (0,1)} \left( \frac{\exp\left(-\frac{px_i}{\varepsilon}\right)}{\varepsilon^3} \right) \right)
\end{aligned}$$

since  $\varepsilon^{-3} > \varepsilon^{-2}$ . Most of the time during analysis, one encounters with exponential terms involving divided by the power function in  $\varepsilon$  which are always the main cause of worry. For their careful consideration while proving the  $\varepsilon$ -uniform convergence, we prove as follows.

**Lemma** For a fixed mesh and for  $\varepsilon \rightarrow 0$ , it holds

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \max_{1 \leq i \leq N-1} \left( \frac{\exp\left(-\frac{px_i}{\varepsilon}\right)}{\varepsilon^m} \right) &= 0, \quad m = 1, 2, 3, \dots \\
\lim_{\varepsilon \rightarrow 0} \max_{1 \leq i \leq N-1} \left( \frac{\exp\left(-\frac{p(1-x_i)}{\varepsilon}\right)}{\varepsilon^m} \right) &= 0, \quad m = 1, 2, 3, \dots
\end{aligned}$$

where  $x_i = ih, h = \frac{1}{N}, i = 1, 2, 3, \dots, N-1$ .

Consider the partition  $[0, 1] = \{0 = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = 1\}$  for the interior grid points, we have

$$\begin{aligned}
\max_{1 \leq i \leq N-1} \frac{\exp\left(-\frac{px_i}{\varepsilon}\right)}{\varepsilon^m} &\leq \frac{\exp\left(-\frac{px_i}{\varepsilon}\right)}{\varepsilon^m} \\
\max_{1 \leq i \leq N-1} \frac{\exp\left(-\frac{p(1-x_i)}{\varepsilon}\right)}{\varepsilon^m} &\leq \frac{\exp\left(-\frac{p(1-x_i)}{\varepsilon}\right)}{\varepsilon^m}
\end{aligned}$$

As  $x_1 = h$  and  $x_{N-1} = 1 - h$ , then by the application of L'Hôpital's rule  $m$  times, we get

$$\lim_{\varepsilon \rightarrow 0} \frac{\exp\left(-\frac{ph}{\varepsilon}\right)}{\varepsilon^m} = \lim_{r=\frac{1}{\varepsilon} \rightarrow \infty} \frac{r^m}{\exp(phr)} = \lim_{r=\frac{1}{\varepsilon} \rightarrow \infty} \frac{m}{(ph)^m \exp(phr)} = 0.$$

Hence, the proof is completed.

**Theorem 4.2** Under the hypothesis of boundedness of the discrete solution (i.e., it satisfies the discrete minimum principle), Lemma 4.4.2 and Theorem 4.1, the discrete solution satisfies the following bound:

$$\sup_{0 \leq \varepsilon \leq 1} \max_i |u_i - U_i| \leq CN^{-1}$$

## 4.4 Numerical Examples and Results

In this section, one example is given to illustrate the numerical method discussed above. The exact solutions of the test problem are not known.

**Definition 4.4:** (Double Mesh Principle): Double-mesh principle is used to calculate the maximum absolute error and rate of convergence of the numerical scheme, when the differential equation has no exact solutions[?]. So the accuracy of their numerical solutions will be computed using double mesh principle. Therefore, we use the double mesh principle to estimate the error and compute the experimental rate of convergence to the computed solution. For any value of N, we put

$$E_\varepsilon^N = \max_{0 \leq i \leq 2N} |U_i^N - U_{2i}^{2N}|,$$

where  $U_i^N$  and  $U_{2i}^{2N}$  are the  $i$ -th and  $2i$ -th components of the numerical solutions on meshes of  $N$  and  $2N$  respectively. We compute the uniform error and the rate of convergence as:

$$E^N = \max_{\varepsilon, \mu} E_\varepsilon^N$$

$$R^N = \log_2 \left( \frac{E^N}{E^{2N}} \right)$$

### Example 4.1

Consider the boundary value problem for the delay differential equation.

$$\begin{cases} \varepsilon u'(x) + \mu(1+x)u'(x) - (4 + \sin x)u(x) + u(x-1) = -e^x, & x \in (0, 2) \\ u(x) = 5 + x, & \text{for } x \in [-1, 0] \\ u(2) = 5 \end{cases}$$

Table 4.1: Maximum absolute error and rate of convergence for different values of  $N$  and  $\varepsilon$  with non-standard fitted operator method for the give Example 4.1

$$\varepsilon \in \left\{ \frac{1}{2^{11+k}}, 1 \leq k \leq 10 \right\}$$

| $\mu$     | Present method              |            |            | Kalaiselvan <i>et al.</i> (2019) |           |           |
|-----------|-----------------------------|------------|------------|----------------------------------|-----------|-----------|
|           | Number of mesh elements $N$ |            |            |                                  |           |           |
|           | 64                          | 128        | 256        | 64                               | 128       | 256       |
| $2^{-22}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-23}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-24}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-25}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-26}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-27}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-28}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-29}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-30}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $2^{-31}$ | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $E^N$     | 6.1892E-03                  | 4.1362E-03 | 2.0865E-03 | 4.756E-02                        | 3.276E-02 | 1.486E-02 |
| $R^N$     | 0.58145                     | 0.98722    |            | 0.53781                          | 1.1405    |           |

Table 4.2: Comparison of Maximum absolute error and rate of convergence for different values of  $\mu$  and  $N$  the with non-standard fitted operator method and fitted mesh method for the give Example 4.1

$$\mu \in \left\{ \frac{1}{2^{5+k}}, 1 \leq k \leq 10 \right\}$$

| $\varepsilon$ | Present method              |            |            | Kalaiselvan <i>et al.</i> (2019) |           |           |
|---------------|-----------------------------|------------|------------|----------------------------------|-----------|-----------|
|               | Number of mesh elements $N$ |            |            |                                  |           |           |
|               | 128                         | 256        | 512        | 128                              | 256       | 512       |
| $2^{-34}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-35}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-36}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-37}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-38}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-39}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-40}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-41}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-42}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $2^{-43}$     | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $E^N$         | 2.9621E-02                  | 5.8327E-03 | 1.8448E-03 | 1.241E-01                        | 9.759E-02 | 6.750E-02 |
| $R^N$         | 2.34438                     | 1.6607     |            | 0.3471                           | 0.5319    |           |

# Chapter 5

## Conclusion and scope for Future work

### 5.1 Conclusion

This study introduce the design of the non-standard finite difference methods (NSFDMs) starts mostly with the concept of exact schemes. A major advantage of having an exact difference scheme for a differential equation is that questions related to the usual considerations of consistency, stability and convergence do not arise. The objective of this project is, therefore, to construct  $\epsilon$ -uniform convergent non-standard finite difference method for solving singularly perturbed two parameter delay differential equation using finite difference method. The convergence analysis and stability of the proposed method will also be established.

### 5.2 SCOPE FOR FUTURE

In this thesis, the design of the non-standard finite difference methods is presented for solving singularly perturbed two parameter delay differential equation using finite difference method. Hence the scheme proposed in this thesis can also be extended solving singularly perturbed two parameter delay differential equation , using non-standard finite difference method problems

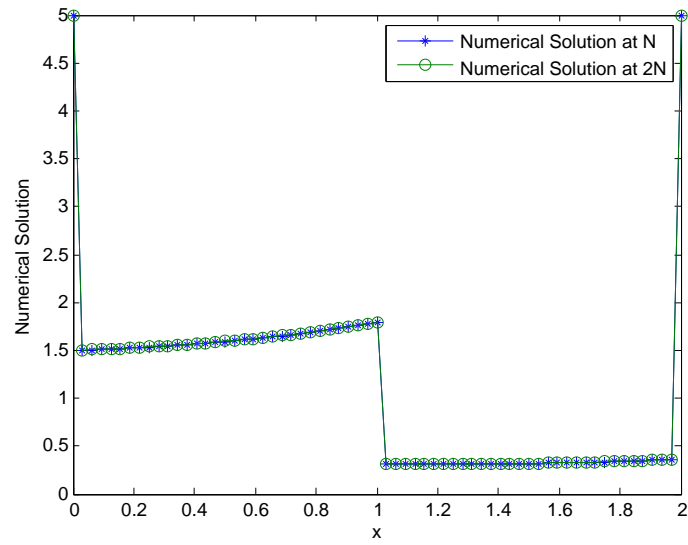


Figure 5.1: The behavior of the Numerical Solution for Example 4.1 at  $\varepsilon = 2^{-34} \rightarrow \varepsilon = 2^{-43}$   $N$  and  $2N$

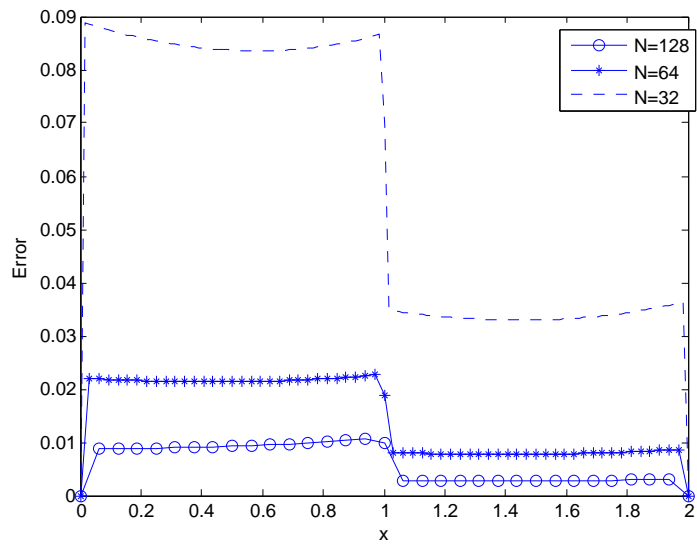


Figure 5.2: The behavior of the Numerical Solution for Example 4.1 at  $\varepsilon = 2^{-32}$  and  $N = 32$

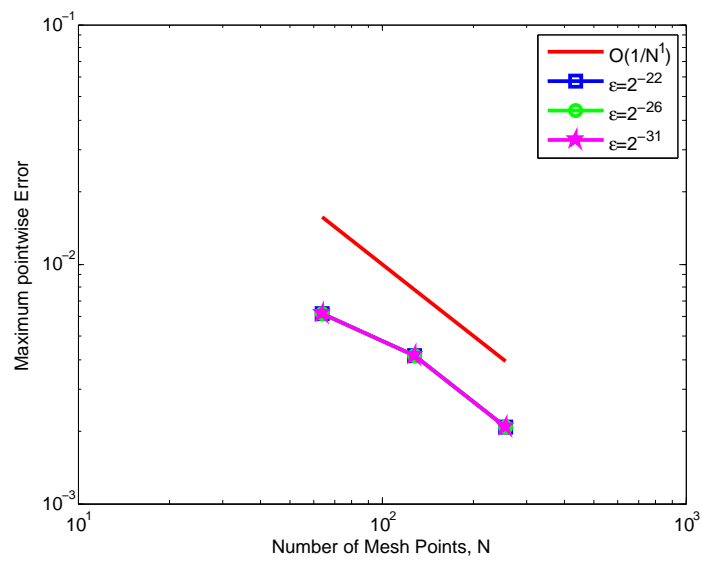


Figure 5.3:  $\varepsilon$ -uniform convergece with fitted operator in log-log scale for Example 4.5.1 at  $\varepsilon=2^{-34}$   $N = 64$ ,  $N = 128$  and  $N = 256$

# References

- [1] Debela, H. G., Duressa, G. F. (2019). Exponentially fitted finite difference method for singularly perturbed delay differential equations with integral boundary condition. *International Journal of Engineering and Applied Sciences*, **11**(4), 476-493.
- [2] Kadalbajoo and Sharma. (2008). A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations. *Applied Mathematics and Computation*, 197 (2008), 692-707
- [3] File, G., Reddy, Y. N. (2013). Domain decomposition method for solving singular perturbation problems. *International Journal of Applied Science and Engineering*, 11(4), 433-448.
- [4] Smith, G. D. (1985). *Numerical solution of partial differential equations: finite difference methods*. Oxford university press.
- [5] Grossmann, C. (2007). *Numerical treatment of partial differential equations*. Springer.
- [6] Subburayan, V., Ramanujam, N. (2012). Asymptotic initial value technique for singularly perturbed convection–diffusion delay problems with boundary and weak interior layers. *Applied Mathematics Letters*, 25(12), 2272-2278.
- [7] Cimen, E. (2018). Numerical solution of a boundary value problem including both delay and boundary layer. *Mathematical Modelling and Analysis*, 23(4), 568-581.
- [8] Miller, J. J., O’riordan, E., Shishkin, G. I. (1996). *Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions*. World scientific..

- [9] Rai, P., Sharma, K. K. (2020). Numerical approximation for a class of singularly perturbed delay differential equations with boundary and interior layer (s). *Numerical Algorithms*, 85(1), 305-328.
- [10] Roos, H. G. (2008). *Robust numerical methods for singularly perturbed differential equations*. Springer.
- [11] Kadalbajoo, M. K., Ramesh, V. P. (2007). Numerical methods on Shishkin mesh for singularly perturbed delay differential equations with a grid adaptation strategy. *Applied Mathematics and Computation*, 188(2), 1816-1831.
- [12] Liu, L. B., Chen, Y. (2014). Maximum norm a posteriori error estimates for a singularly perturbed differential difference equation with small delay. *Applied Mathematics and Computation*, 227, 801-810.
- [13] Geng, F. Z., Qian, S. (2015). Modified reproducing kernel method for singularly perturbed boundary value problems with a delay. *Applied Mathematical Modelling*, 39(18), 5592-5597
- [14] Patidar, K. C., Sharma, K. K. (2006). Uniformly convergent nonstandard finite difference methods for singularly perturbed differential-difference equations with delay and advance. *International Journal for Numerical Methods in Engineering*, 66(2), 272-296.
- [15] Duressa, G. F., Debela, H. G. (2021). Numerical solution of singularly perturbed differential difference equations with mixed parameters. *Journal of Mathematical Modeling*, 9(4), 691-705.
- [16] Kumar, D., Kadalbajoo, M. K. (2011). A parameter-uniform numerical method for time-dependent singularly perturbed differential-difference equations. *Applied Mathematical Modelling*, 35(6), 2805-2819.
- [17] Gobena, W. T., Duressa, G. F. (2021). Parameter uniform numerical methods for singularly perturbed delay parabolic differential equations with non-local boundary condition. *International Journal of Engineering, Science and Technology*, 13(2), 57-71.

- [18] Debela, H. G., Duressa, G. F. (2019). Exponentially fitted finite difference method for singularly perturbed delay differential equations with integral boundary condition. *International Journal of Engineering and Applied Sciences*, 11(4), 476-493.
- [19] Ranjan, R., Prasad, H. S. (2021). A novel approach for the numerical approximation to the solution of singularly perturbed differential-difference equations with small shifts. *Journal of Applied Mathematics and Computing*, 65(1-2), 403-427.
- [20] Patidar, K. C. (2005). On the use of nonstandard finite difference methods. *Journal of Difference Equations and Applications*, 11(8), 735-758.
- [21] Cakir, M. (2016). A numerical study on the difference solution of singularly perturbed semilinear problem with integral boundary condition. *Mathematical Modelling and Analysis*, 21(5), 644-658.
- [22] Cimen, E., Cakir, M. (2017). A priori estimates for solution of singularly perturbed boundary value problem with delay in convection term. *J. Math. Anal*, 8(1), 202-211.
- [23] Selvi, P. A., Ramanujam, N. (2016). An iterative numerical method for singularly perturbed reaction–diffusion equations with negative shift. *Journal of Computational and Applied Mathematics*, 296, 10-23.
- [24] Zarin, H. (2014). On discontinuous Galerkin finite element method for singularly perturbed delay differential equations. *Applied Mathematics Letters*, 38, 27-32.
- [25] Mohapatra, J., Natesan, S. (2011). Uniformly convergent numerical method for singularly perturbed differential-difference equation using grid equidistribution. *International Journal for Numerical Methods in Biomedical Engineering*, 27(9), 1427-1445.
- [26] Rai, P., Sharma, K. K. (2012). Fitted mesh numerical method for singularly perturbed delay differential turning point problems ex
- [27] Podila, P. C., Kumar, K. (2019). A class of finite difference schemes for singularly perturbed delay differential equations of second order. *Turkish Journal of Mathematics*, 43(3), 1061-1079.