CONFORMABLE REDUCED DIFFERENTIAL TRANSFORM METHOD FOR FRACTIONAL ALLEN CAHN EQUATION

A THESIS SUBMITTED TO DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, JIMMA UNIVERSITY, FOR PARTIAL FULFILMENT OF THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCES(M.SC.) IN MATHEMATICS.

BY: BURTUKAN FIROMSA NURGI
ADVISOR: YESUF OBSIE (Ph.D.)
CO-ADVISOR: MELKAMU GOSHINA (M.Sc.)

JUNE, 2024
JIMMA, ETHIOPIA

DECLARATION

This is to certify that this thesis entitled "conformable reduced differential transform method for time fractional Allen-Cahn equation" submitted in partial fulfillment of the requirement for the award of the Degree of Masters of Science (M.Sc.) in Differential Equation to the school of graduate studies, Jimma University, through the Department of Mathematics studies conducted by Mr. Burtukan Firomsa.

To best of my knowledge and beliefs, the substances included under this thesis work has not been submitted for the fulfillment of any award qualification in earlier time.

Name of student: Burtukan Firomsa Nurgi

Signature: -----------------------Date: ---------------------------

Name of Advisor: Yesuf Obsie (Ph.D.)

Signature: ------------------------ Date: -------------------------

Name of co-Advisor: Melkamu Goshina (M.Sc.)

Signature: ------------------------ Date: ---------------------

ACKNOWLEDGMENT

First of all, I am indebted to my almighty God, who gave me a long life and helped me pass through different ups and downs to reach this point. Next, I would like to express my deepest gratitude to my advisor Yesuf Obsie (Ph.D.) and co-advisor Melkamu Goshina (M.Sc.) for their unreserved support, unlimited advice, and constructive and critical comments throughout the preparation of this thesis. Finally, I would like to thank Jimma University for providing me with these opportunities.

ACRONYMS AND ABBREVIATION

ACE - Allen-Cahn equation

CRDTM - Conformable Reduced Differential Transform Method

DTM - Differential transform method

FC -Fractional Calculus

FDE - Fractional Differential Equation

FKDV - Fifth-order korteweg-de Vries

FPDEs - Fractional Partial Differential Equations

HPM - Homotopy Perturbation Method

IVPs - Initial Value Problems

PDEs - partial differential equation

VIM - Variation Iteration Method

ABSTRACT

In this manuscript, the Conformable Reduced Differential Transform Method was applied to obtain an approximate analytical solution of one dimensional time fractional Allen-Cahn equation subject to initial condition. This method provides an approximate analytical solution in the form of a convergent series. The scheme was tested through supportive examples and the obtained results were compared with previous works. Solutions were presented in graphical form and results indicate that the Conformable Reduced Differential Transform Method is more accurate, efficient, and convenient for solving one dimensional time fractional Allen-Cahn equation.

Key Words: Reduced Differential Transform Method, Conformable Reduced Differential Transform Method, and Fractional Allen-Cahn Equation.

Table of Contents

LIST OF TABLES

LIST OF FIGURES

CHAPTER ONE
INTRODUCTION

1.1. Background of the study

Fractional calculus(FC) is a branch of mathematical analysis that studies the possibility of taking real numbers as powers of the differential operator and the integration operator(Gubes, 2013). FC has been increasingly used in many research areas in recent years. Fractional Differential Equations efficiently map natural evolutions related to energy transfer, relaxation models and thermos elasticity in viscoelasticity, porous electrode models, thermal stress, electromagnetics and viscous dissipation systems(Zada et al., 2022).

Most mathematical models are derived from real-world problems that can be modelled by differential equations of integer or fractional order. The idea of fractional derivatives was first described in the 17^{th} century by the great mathematicians Newton and Leibniz (Jneid & Chaouk, 2020) has been exploited by numerous applications in complex non-linear system, manifesting itself in variety of important phenomena such as fluid dynamics, damping laws and electricity and received widespread attention. Fractional derivatives provide more accurate models of real-world problems compared to integer derivative(Singh & Kumar, 2017).

A fractional order differential equation have been successfully applied to problems in a variety of scientific fields, including biology, economics, polymer chemistry, mechanics, aerodynamics, control theory and biophysics (Hilfer, 2000;Kilbas et al., 2006). Fractional derivatives provide an excellent instrument for the descriptive and hereditary properties of various materials and processes(Kempfle et al., 2002). So, solving FPDEs was completely important in the circumstance of Applied Mathematics, theoretical Physics and Engineering Sciences(Sohail&Mohyud-din,2012).

Mathematical approaches to partial differential equations are divided in to two methods called Analytical methods which strive to find exact formulae for the dependent variable as a function of independent variables and numerical methods which result in approximate values of dependent variable at prescribed and discrete location within a finite domain of the independent variables (Richard B, 2018).

But there are mathematical approaches which can be neither of the two methods. These are semi analytical and semi numerical methods. For example, reduced differential transform method (RDTM) is semi analytical method and used to find exact solutions or closed approximate solution of a differential equation (Srivastava et al .,2014). It was an iterative procedure for obtaining Taylor series solution of differential equation(Sohail & Mohyud-din, 2012).In this work, we apply the conformable fractional reduced differential transform method to the time-fractional Allen-Cahn equation of the form:

$$\partial_t^{\alpha} u(x,t) - u_{xx}(x,t) + u^3(x,t) - u(x,t) = 0, \quad (1.1)$$

with initial condition $u(x,0) = h(x)$ (1.2)

which was obtained from the general nonlinear Diffusion equation

$$\frac{du}{dt} = \delta \frac{d^2u}{dx^2} + \alpha u + \beta u^n.$$

Where u is concentration,δ is diffusion coefficient,$\alpha = 1$and $\beta = -1$are real constant and n = 3, which has various applications in quantum mechanics, biology and plasma physics as in (Ahmad et al., 2020);

In equation (1.1) operator for ∂_t^{α} denotes the Caputo type fractional derivative of order α where $\alpha \in (0,1]$ with respect to time and initial condition.

Recently, Allen-Cahn equation was solved by (Kim et al., 2021) using finite difference method and (Yang et al., 2023),used some other numerical method like finite element method, Fourier special method and other to solve time fractional Allen-Cahn equation. However, conformable reduced DTM is not applied to solve time fractional Allen-Cahn Equation yet, which make this present study valuable and novel.

1.2. Statement of the problem

Allen-Cahn equation is a simple mathematical model for certain phase separation process. It also serves as prototypical example for semi-linear parabolic partial differential equations. And also, Allen-Cahn equation can be used in a wide variety of engineering and mathematical physics applications. Solving fractional Allen-Cahn equations by applying reduced differential transform method was not presumably presented in the existing literature. The advantage of using FRDTM is it can be applied without any linearization or discretization or restrictive assumptions and it is very reliable, efficient, and effective powerful computational technique for solving physical problems(MekonnenYadeta & Obsie, 2020).

Therefore, the aim of this study was to use the Conformable Fractional Reduced Differential Transform Method to find an approximate analytical solution for time fractional Allen-Cahn equation. As a result, this study mainly focused on the following basic points.

- Time fractional Allen-Cahn equation.
- Conformable fractional derivative.
- Reduced Differential Transform Method.

1.3. Objectives of the study

1.3.1. General objective

The general objective of the study was to find approximate analytic solution of time fractional Allen-Cahn equation by using conformable time fractional reduced differential transform method (CTRDTM).

1.3.2. Specific objectives

The specific objectives of the study were**:**

- To define the governing equation (time fractional Allen-Cahn equation).
- To define Conformable derivative and fractional RDTM.
- To apply Conformable reduced differential transform method on TFACE.
- To prove the convergence of the proposed method.
- To verify the applicability of the methods using model examples.

1.4 The significance of the Study

This research was considered of vital importance for the following reasons.

- ➢ It develops the researcher skill on mathematical (applied) research.
- ➢ It provides techniques for solving time fractional ACE by using RDT method.
- ➢ It familiarizes researcher with the scientific communication in mathematics.
- ➢ It was help as a reference material for anyone who was work on this area.

1.5 Delimitation of the study.

The study was delimited to solve time fractional Allen-Cahn equation using conformable reduced differential transform method.

CHAPTER TWO

LITERATURE REVIEW

Fractional differential equations are located in the centre of numerous research area in recent years because of their commonly appearance in different kinds of applied sciences especially in the fields of mathematics and engineering sciences. Many natural phenomena can be presented by boundary value problems of fractional differential equations. Many authors in various fields such as chemical physics, fluid flows, electrical networks, viscoelasticity, have tried to present a model of these phenomena by fractional differential equations(Oldham & Spanier, 1974 ; B. Ross,1975).

Some authors such as (Podlubny.,1999; Schneider & Wyss, 1989;Beyer & Kempfle, 1995; Mainardi, 1996) discussed fractional order and partial differential equations. And as this new fractional calculus came to light, many fractional partial differential equations (FPDEs) unfolded in the process. Furthermore, FPDEs turned out to play a substantial role in modelling countless phenomena that arise in applied sciences. Fractional derivative can be defined in different ways such as Riemann Liouville, Hadamard, Coimbra, Caputo, Riesz Marchaud and Canavati (Machado et al., 2011). The two widely used are Riemann-Livoulle and Caputo definition of fractional derivative(Jneid & Chaouk,2020).

The advantage of using fractional models of differential equations in physical models is due to their non-local property. Fractional order derivative is non-local while integer order derivative is local in nature. It shows that the upcoming state of physical system also dependens on all of its historical states in addition to its present state. Hence the fractional models are more realistic and fractional derivatives are often used to model problems in acoustics, fluid mechanics, diffusion, electromagnetism, signal processing, biology, finance and some more process(Prakash et al., 2019).The Allen-Cahn equation is a simple mathematical model for certain phase separation processes. It is a second order semi-linear parabolic partial differential equation representing some natural physical phenomenon.

The model attracted attention of many researchers around the world due to its applicability in many fields of science and technology. The principal role of this model is to describe the process of phase separation in iron alloys, including order-disorder transition. Some researchers have extended this model to the scope of fractional calculus using the Caputo and Riemann-Liouville fractional derivative(Jefain & Algahtani, 2016; Hussain et al., 2019) .

The Allen-Cahn equation has been extensively used to study various physical problems, such as crystal growth, image segmentation and the motion by mean curvature flows (Hussain et al., 2019) . In particular, it has become a basic model equation for the diffuse interface approach developed to study phase transitions and interfacial dynamics in material science. Thus, an efficient and accurate method for the solution of this equation has practical significance and has drawn the attention of many researchers(Jefain & Algahtani, 2016; Hussain et al., 2019).

Esen et al. (2013) obtained the approximate analytical solution of time fractional damped burger and Allen-Cahn equations by means of the Homotopy analysis method (HAM). In the model problems, an appropriate choice of the auxiliary parameter has been examined for increasing values of time. Wang et al. (2022) numerically studied the time fractional Allen-Cahn equation.

The local discontinuous Galerkin method is applied to discretize the spatial derivative. With help of the discrete fractional Kronwall-type in equalities, the stability and optimal error estimates of the full discrete numerical schemes are demonstrated. Khalid et al.(2020) presented the collocation approach based on redefined cubic B-spline functions and finite difference formulation to study the approximate solution of time fractional Allen-Cahn equation. They discretized the time fractional derivative of order $\alpha \in (0,1]$ by using finite forward difference formula and bring functions into action for spatial discretization.

Kanth et al. (2021) analysed the time fractional Klein-Gordon equation using the natural transform decomposition method and the iterative Shehu transform method with singular kernel derivative. They employed natural transform and Shehu transform, followed by inverse natural transform and inverse Shehu transform respectively, to obtain the solution.

Before the nineteenth century, there was no scheme available for the analytical solutions of the fractional differential equations. In the beginning of the twentieth century, researchers started to pay attention to find the robust and stable analytical approaches for the exact (approximate) solution of the fractional differential equations (Tamsir & Srivastava, 2016).

Subsequently, several schemes such as the Adomain decomposition methods (Ray & Bera, 2005), differential transform method (Odibat et al., 2008), Variation iteration method (Safari et al., 2009), Homotopy perturbation method (Zhang et al., 2014), Local fractional variation iteration method (X. J. Yang et al., 2014), and Shifted Chebyshev polynomials based method (Karunakar & Chakraverty, 2019), have been developed for the analytical solutions of fractional differential equations(Desalegn et al., 2020).

In 1986, an up dated version of Taylor series method, called the differential transform method (DTM) was introduced by Zhou Jk (1986), and then applied DTM in order to solve electric circuit problem. In past several decades, many authors mainly had paid attention to study the solution of fractional differential equations by using various developed method such as RDTM, VIM, DTM, ADM, Tanh-Coth method, and Sine-Cosine method. Among of these VIM, DTM, ADM, Tanh-coth method, and sine-cosine method were used to solve non-linear partial differential equations (Sohail & Mohyud-din, 2012).

Most of the above methods sometimes require complex and huge calculation in order to obtain approximate solutions. To overcome such difficulties and drawbacks, Keskin & Oturanc developed an alternative method called fractional reduced differential transform method and used to solve linear and non-linear wave equations and they showed the effectiveness and accuracy of the proposed method(Keskin & Oturanc,2010).

Later on, researchers have applied the Fractional Reduced Differential Transform Method (FRDTM) successfully to obtain analytic solution of differential equations. For example: Mahmoud Rawashden(2013), used the FRDTM to find exact and approximate solution for Garden equation, Variant Non-linear Water Wave equation and the Fifth-order korteweg-de Vries equation. İbis & Bayram(2012),used the FRDTM to approximate the solution of Drinefel'd- sokolov- Wilson equations, Coupled Burgers equations and modified Boussinesq equation.

Saravanan & Magesha(2013) used the RDTM and ADM to solve analytic solution for linear and non-linear Newell-White head- Segel equation. Srivastava et al.(2015) used the RDTM to obtain analytic solution of telegraph equation. Murat Gubes &Richard B (2018), used the RDTM to obtain analytic solution for non-linear time-dependent Foam Drainage equations.

Recently, the well-known method named FRDTM has been widely used by a huge number of studies for solving such kind of equations. For instance, (Jafari et al.,2016;Thabet & Kendre, 2016; Jneid & Chaouk ,2017 ; Acan et al., 2017; Singh & Kumar, 2018; Yang et al., 2018; Murat Gubes &Richard B. 2018; Chaouk & Jneid 2019) obtained solution of FPDEs by using RDTM (Chaouk & Jneid 2019).

MekonnenYadeta & Obsie (2020), have used TFRDTM to approximate analytical solution of one-dimensional Beam equations and conclude that the method is very simple to utilize and brevity of calculations(MekonnenYadeta & Obsie, 2020).

Motivated by the above discussion in this study we are going to apply CRDTM to solve Time Fractional Allen-Cahn equation:

$$\partial_t^{\alpha} u(x,t) - u_{xx}(x,t) + u^3(x,t) - u(x,t) = 0, \alpha \in (0,1],$$

with initial condition $u(x,0) = h(x)$.

This is obtained from the general non-linear Diffusion equation

$$\frac{du}{dt} = \delta \frac{d^2u}{dx^2} + \alpha u + \beta u^n, \delta = 1 = \alpha, \beta = -1.$$

CHAPTER THREE

METHODOLOGY

3.1. Study Area and period

This study was conducted in Jimma University, College of Natural Sciences, Department of Mathematics from September, 2023 to June, 2024.

3.2 Study Design

The design of the study was analytical.

3.3. Source of Information

The sources of data for this study were secondary data which can be collected through reference books, internet, reading online books and different published research articles or journals related to the topic.

3.4 Mathematical Procedures

For the success of the objectives of the study, the following procedures were followed.

Step 1: Applying CRDTM on the governing equation to get the transformed function.

Step 2: Use $U_0^{\alpha}(x) = f(x)$ and plugging it in step 1, to simplify the transformed function.

Step 3: Evaluating the $(k+1)^{th}$ terms obtained in step 2, to get $U_k^{\alpha}(x),\ k = 1,2,3, \dots, n$.

Step 4: Applying inverse CRDT on step 3, to get the approximate solution in

the form: $u_n(x,t) = \sum_{k=0}^{n} U_k^{\alpha}(x) t^{k\alpha}$.

Step 5: Testing the convergence of the solution, $u_n(x,t), n = 0,1,2, \dots$ obtained in step 4,

to obtain the desired approximate analytical solution and apply the method

to illustrative examples.

CHAPTER FOUR
PRELIMINARIES

4.1. Fractional Calculus

This section presents the basic definitions, theorems and properties of conformable fractional reduced differential transform method and fractional calculus that are relevant for this research work.

Definition 4.1 (Petras, 2009).

Euler's Gamma function, which is defined as:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx. \tag{4.1}$$

This function is generalization of a factorial in the following form:

$$\Gamma(n) = (n-1)!$$

Definition 4.2 (Podlubny, 1999) A real function $u(t), t > 0$ is said to be in space $C_{\mu,}$ $\mu \in R$ if there exists a real number $\rho > \mu$ such that $u(t) = t^p u_1(t)$, where $u_1(t) \in C(0,\infty)$, and it is said to be in the space if and only if $u^{(n)} \left(u^{(n)} = \frac{d^n}{dt^n} u(t) \in C_\mu\ n \in N\right)$.

4.2. Fractional Derivatives

The two most commonly used fractional derivatives, namely the Riemann-Liouville and Caputo are defined as follows (Khalil et al., 2014).

Definition 4.3 (Miller et al., 1993)

i) Riemann-Liouville definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\,dx. \tag{4.2}$$

ii) Caputo definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}}\,dx. \tag{4.3}$$

Note: The Caputo fractional derivatives allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations.

Definition 4.4 (Khalil, et al., 2014) Let $f(t)$ be n times differentiable function. Then the conformable fractional derivative $f(t)$ of order α is defined by

$$T_\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f^{(\lceil\alpha\rceil-1)}\left(t+\epsilon t^{(\lceil\alpha\rceil-\alpha)}\right)-f^{(\lceil\alpha\rceil-1)}(t)}{\epsilon}, \tag{4.4}$$

for all $\mathrm{t} > 0, \alpha \in (\mathrm{n}-1, \mathrm{n}], \lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Lemma4.1 let u be an n times differentiable function at of x.Then $f_\alpha u(x) = x^{\lceil\alpha\rceil-\alpha} u^{(\lceil\alpha\rceil)}(x)$, $\forall \mathrm{x} > 0, \alpha \in (n, n+1]$.

Definition 4.5 (Khalil et al., 2014;Jefain & Algahtani,2016). Given a function f: [0,∞) → R. Then the conformable fractional derivative of f of order α is defined by:

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}, \text{ for all } \mathrm{t} > 0,\ \alpha \in (0,1]. \tag{4.5}$$

Lemma4.2(Khalil et al., 2014) Let $\alpha \epsilon (0,1]$ and f, g be α- differentiable at a point $t > 0$.

i)$T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.

ii) $T_\alpha(t^p) = \mathrm{p}t^{p-\alpha}$, for all $p \in \mathbb{R}$.

iii) $T_\alpha(\lambda) = 0$, for all constant function $f(t) = \lambda$.

iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

v) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.

vi) If f is differentiable, then$T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Proof. Parts (i) through (iii) follow directly from the definition. We choose to prove (iv) and (vi) only since they are crucial. Now, for fixed t > 0,

$$\begin{aligned}
T_\alpha(fg)(t) &= \lim_{\varepsilon\to 0} \frac{f\left(t+\epsilon t^{1-\alpha}\right)g\left(t+\varepsilon t^{1-\alpha}\right)-f(t)g(t)}{\epsilon} \\
&= \lim_{\varepsilon\to 0} \frac{f\left(t+\epsilon t^{1-\alpha}\right)g\left(t+\varepsilon t^{1-\alpha}\right)-f(t)g\left(t+\varepsilon t^{1-\alpha}\right)+f(t)g\left(t+\varepsilon t^{1-\alpha}\right)-f(t)g(t)}{\epsilon} \\
&= \lim_{\varepsilon\to 0}\left(\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\epsilon}. g(t+\varepsilon t^{1-\alpha})\right) + f(t)\lim_{\varepsilon\to 0}\frac{g\left(t+\varepsilon t^{1-\alpha}\right)-g(t)}{\epsilon} \\
&= T_\alpha(f)(t)\lim_{\varepsilon\to 0} g(t+\varepsilon t^{1-\alpha}) + f(t)T_\alpha g(t).
\end{aligned}$$

Since g is continuous at t, $\lim_{\varepsilon\to 0} g(t+\varepsilon t^{1-\alpha}) = g(t)$ .This completes the proof of part (iv).

(v) can be proved in a similar way. To prove (vi), let $h = \varepsilon t^{\alpha-1}$ in definition 4.6, and then $\varepsilon = t^{\alpha-1}h$.

Therefore, $$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

$$= \lim_{h\to 0} \frac{f(t+h)-f(t)}{t^{\alpha-1}h}$$

$$= t^{\alpha-1} \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$$

$$= t^{\alpha-1} \frac{df}{dt}(t).$$

The other properties$(\mathrm{i} - \mathrm{v})$ can be proved in similar manner.

4.3 Reduced Differential Transform Method

The RDTM was first proposed by Turkish mathematician Keskin in 2009. It has received much attention of many authors since; it has applied to solve a wide variety of problem (Reza & Malek, 2013), and defined as follows:

Definition4.6 (Rawashdeh, 2017) If $u(x,t)$ is analytic and differentiable continuously with respect to time t and space x in the domain of interests then:

$$U_k(\mathrm{x.t}) = \frac{1}{k!}[\frac{\partial^k}{\partial x^k}\mathrm{u(x,t)}]_{t=t_0}\ , \tag{4.6}$$

where the t- dimensional spectrum functions, $U_k(x)$ is the transformed function. The lowercase $\mathrm{u(x,t)}$ represents the original function. The differential inverse transforms of $U_k(x)$ is defined as follows:

$$\mathrm{u(x,t)} = \sum_{k=0}^{\infty} U_k\,(x)(t-t_o)^k, \tag{4.7}$$

using (4.6) in (4.7), we get

$$\mathrm{u(x,t)}\ =\ \sum_{k=0}^{\infty}\frac{1}{k!}[\frac{\partial^k}{\partial x^k}\mathrm{u(x)}]_{t=t_0}(\mathrm{t}-t_0)^k. \tag{4.8}$$

Theorem 4.1 (Deresse, 2022) If $\mathrm{w(x,t)} = u^2(x,t)$, then $w_k^\alpha(x) = \sum_{r=0}^{k} U_r^k(x)$.

Proof with the help of definition $\mathrm{u(x,t)}$can be written as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k^\alpha(x)(t-t_0)^{k\alpha}. \tag{4.9}$$

Then,$w(x,t)$ is obtained as

$$w(x,t) = u^2(x,t) = \sum_{k=0}^{\infty} U_k^\alpha(x)(t-t_0)^{k\alpha} \sum_{k=0}^{\infty} U_k^\alpha(x)(t-t_0)^{K\alpha}$$

$$= (U_0^\alpha(x) + U_1^\alpha(x)(t-t_0)^\alpha + U_2^\alpha(x)(t-t_0)^{2\alpha} + U_3^\alpha(x)(t-t_0)^{3\alpha} + U_4^\alpha(x)(t-t_0)^{4\alpha} + \cdots) \times (U_0^\alpha(x) + U_1^\alpha(x)(t-t_0)^\alpha + U_2^\alpha(x)(t-t_0)^{2\alpha} + U_3^\alpha(x)(t-t_0)^{3\alpha} + U_4^\alpha(x)(t-t_0)^{4\alpha} \ldots)$$

$$= U_0^\alpha(x)U_0^\alpha(x) + U_0^\alpha(x)U_1^\alpha(x) + U_1^\alpha(x)U_0^\alpha(x))(t-t_0)^\alpha + (U_0^\alpha(x)U_2^\alpha(x) + U_1^\alpha(x)U_1^\alpha(x) + U_2^\alpha(x)U_0^\alpha(x)(t-t_0)^{2\alpha} + \cdots$$

$$= \sum_{k=0}^{\infty} \sum_{r=0}^{k} U_{r=0}^\alpha\,(x)U_{k-r}^\alpha(x)(t-t_0)^{k\alpha}.$$

Lemma4.3 (Deresse, 2022) If $w(x,t) = u^3(x.t)$, then
$w_k^\alpha(x) = \sum_{r=0}^{k} \sum_{i=0}^{r} U_i^\alpha(x) U_{r-i}^\alpha U_{k-r}^\alpha(x)$.

Proof Observe that $u^3(x.t) = u^2(x,t)u(x,t)$.

$$w_k^\alpha(x) = u^3(x.t) = \frac{1}{\alpha^k k!}[(\partial_t^\alpha)^k u^3(x,t)]_{t=0}$$

$$= \frac{1}{\alpha^k k!}[(\partial_t^\alpha)^k u^2(x,t)u(x,t)]_{t=0}$$

$$= \sum_{r=0}^{k} F_r^\alpha(x) U_{k-r}^\alpha\,(x), \tag{4.10}$$

where $F_r^\alpha(x)$ is the reduced differential transform of $u^2(x,t)$; i.e., by using theorem (4.1)

$$F_r^\alpha(x) = \sum_{i=0}^{r} U_i^\alpha(x) U_{r-i}^\alpha(x). \tag{4.11}$$

Thus, $w_K^\alpha(x) = \sum_{r=0}^{K} F_r^\alpha(x) U_{k-r}^\alpha(x) = \sum_{r=0}^{k} \sum_{i=0}^{r} U_i^\alpha(x) U_{r-i}^\alpha(x) U_{k-r}^\alpha(x)$.

4.3.1 Conformable Fractional Reduced Differential Transform Method

Definition 4.7 (Acan et al., 2016) Assume $u(x,t)$ is analytic and continuously differentiable with respect to time t and space x in its domain. Then the CFRDT of $u(x,t)$ is defined as:

$$U_k^\alpha(x) = \frac{1}{\alpha^k k!}[\left({}_tT_\alpha^{(k)} u\right)]_{t=t_o}, \tag{4.12}$$

where $0 < \alpha \leq 1$, α a parameter describing the order of conformable fractional derivative,

${}_tT_\alpha^{(k)}u = \underbrace{({}_tT_\alpha \, {}_tT_\alpha \dots\dots {}_tT_\alpha)}_{k\ times} u(x,t)$ and t- dimensional spectrum functions $U_k^\alpha(x)$ is the CFRDT function and k is a non-negative integer.

Definition 4.8 Let $U_k^\alpha(x)$ be the CFRDT of $u(x,t)$ .Then inverse conformable fractional reduced differential transform of $U_K^\alpha(x)$ is defined(Acan et al., 2016) as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k^\alpha(x)\,(t - t_o)^{\alpha k}, \tag{4.13}$$

combining equations (4.12) and (4.13), we get

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\alpha^k k!}\,[{}_tT_\alpha^{(k)}u\,(x,t)]_{t=t_o}\,(t - t_o)^{\alpha k}. \tag{4.14}$$

Conformable fractional reduced differential transform of initial conditions for integer order derivatives are defined as:

$$U_k^\alpha(x) = \begin{cases} \frac{1}{(\alpha k)!}\,[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x,t)]_{t=to} \;, & if\ \alpha k \in \mathbb{Z}^+ \\ 0\ if\ \alpha k \notin \mathbb{Z}^+. \end{cases} \text{, for k =1, 2, 3… } (\frac{n}{\alpha} - 1) \tag{4.15}$$

where n is the order of conformable fractional PDE.

By considerations of $U_o^\alpha(x) = u(x)$ as the transformed initial condition, $u(x,0) = u(x)$.

After straight forward iterative calculation the $U_k^\alpha\,(x)$ values obtain for $k = 1, 2, 3, \dots n$.

In real applications, the function $u(x,t)$ is represented by a finite series of equ. (4.8) around $t_o = 0$ and can be written as:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{\infty} U_k^a(x) t^{ak} + \text{Ɍ}_n(x,t), \tag{4.17}$$

where $\text{Ɍ}_n(x,t) = \sum_{k=n+1}^{\infty} U_k^a(x) t^{ak}$ is negligibly small.

Furthermore, the conformable fractional reduced differential inverse transform of the set

$\{U_k^a(x)\}_{k=0}^n$ gives the approximation solution as:

$\tilde{u}_n(x,t) = \sum_{k=0}^{\infty} U_k^a\,(x) t^{ak}$, where n is the order of the approximation solution.

The exact solution is given by

$$\mathrm{u(x,t)} = \lim_{\infty} \tilde{u}_{n(x,t)} = \sum_{k=0}^{n} U_k^a\,(x) t^{ak} \tag{4.18}$$

$$\mathrm{u(x,t)} = U_0^a(x) + U_1(x)t^a + U_2^a(x)t^{2a} + U_3^a(x)t^{3a} + \cdots$$

Theorem 4.2 (Thabet & Kendre, 2018)Assume that u is infinitely α- differentiable function for some $0 < \alpha \leq 1$ at a point t_0 . Then the fractional power series expansion as

$$\mathrm{u\ (x,t)} = \sum_{k=0}^{\infty} \frac{({}_tT_\alpha^{(k)}u)(t_o)(\mathrm{t} - t_o)^{ak}}{a^k k!}\ , t_o < t < t_o + R^{\frac{1}{a}}, \mathrm{R} > 0.$$

Here, $({}_tT_\alpha^{(k)}u)(t_o)$ denotes the application of the fractional derivative k times.

For instance, let $\mathrm{y(t)} = e^{\frac{t^a}{a}}$ and $t_o = 0$ then $y(t) = e^{\frac{t^a}{a}} = \sum_{k=0}^{\infty} \frac{t^{ka}}{a^k K!}$

For $y(t) = \frac{1}{1-\frac{t^a}{a}}$, power series representation is $y(t) = \frac{1}{1-\frac{t^a}{a}} = \sum_{k=0}^{\infty} \frac{t^{ka}}{a^k}, t\epsilon[0,1)$.

Theorem 4.3 (Acan et al., 2016) If $\mathrm{u\ (x,t) = v(x\,,t) \pm w(x,t)}$ then $U_k^a(x) = V_k^a \pm W_k^a(x)$.

Proof: - Suppose that $U_k^a(x), V_k^a(\mathrm{x})$ and $W_k^a(x)$ are the t-dimensional spectrum functions transformed function of $\mathrm{u\ (x\,,t\,), v\ (x\,,t\,)}$& $\mathrm{w\ (x\,,t\,)}$ respectively.

we want to show that $U_k^a(x) = V_k^a(x) \pm W_k^a(x)\ldots$

Appling conformable fractional reduced differential transform on both sides, we get

$\mathrm{u(x,t) = \ v\ (x,t\,)\ \pm\ w\ (x\,,t)}$, we have conformable fractional reduced differential transform

$$\mathrm{u(x,t)] = \ CFRDT\ [v\ (x,t) \pm\ CFRDT\ [w(x,t)].} \tag{4.19}$$

Using the theorem 4.1 and by definition 4.8, we obtain

Conformable fractional reduced differential transform $[\mathrm{u(x,t)}] = \frac{1}{a^k k!}\left[({}_tT_\alpha^{(k)}u)\right]_{t=t_o}$.

Conformable fractional reduced differential transform $[\mathrm{v(x,t)}] = \frac{1}{a^k k!}\ [({}_tT_\alpha^{(k)}v)]_{t=to}$ and

Conformable fractional reduced differential transforms $[\mathrm{w(x,t)}] \frac{1}{a^k k!} [({}_tT_\alpha^{(k)}w)]_{t=to}$, where

k = 1, 2, 3 (4.20)

Now substituting (4.20) in to (4.19), we get

$$\frac{1}{a^k k!}\left[\left({}_tT_\alpha^{(k)}\, u\right)\right]_{t=to} = \frac{1}{a^k k!}[({}_tT_\alpha^{(k)}\, v)]_{t=to} \pm \frac{1}{a^k k!}\left[\left({}_tT_\alpha^{(k)}\, w\right)\right]_{t=to}. \quad (4.21)$$

Also, by definition 4.7 we write

$U_k^a(x) = \frac{1}{a^k k!}[({}_tT_\alpha^{(k)} u)]_{t=to}$, $V_k^a(x) = \frac{1}{a^k k!}[({}_tT_\alpha^{(k)} v)]_{t=to}$ and

$$W_k^a(\mathrm{x}) = \frac{1}{a^k k!}\left[\left({}_tT_\alpha^{(k)} w\right)\right]_{t=to}. \quad (4.22)$$

Substituting (4.18) into (4.17) yields

$U_k^a(x) = V_k^a(x) \pm W_k^a(\mathrm{x})$, k = 0, 1, 2 ……and the proof is completed

Theorem 4.4 Let c be a constant, if $\mathrm{u(x,t)} = \mathrm{cv(x,t)}$, then $U_k^a(x) = cV_k^a\,(x)$.

Proof: - Conformable fractional reduced differential transform of v(x, t) that

$V_k^a(x) = \frac{1}{a^k k!}[({}_tT_\alpha^{(k)}\, v)]_{t=to}$ by the help of theorem 4.1 we can write

$$U_k^a(x) = \frac{1}{a^k k!}[({}_tT_\alpha^{(k)}\, cv)]_{t=to}$$

$= \frac{c}{a^k k!}\left[\left({}_tT_\alpha^{(k)}\, v\right)\right]_{t=to} = cV_K^a(x)$. This completes the proof of the theorem.

Theorem 4.5 (Thabet & Kendre, 2018) If $\mathrm{u(x,t)} = \mathrm{v\,(x,t)\,w(x,t)}$,then,

$$U_k^a(x) = \sum_{s=0}^{k} V_s^k(x)\, W_{k-s}^a(x).$$

Proof: - with the help of definition 4.8, v(x, t) and w (x, t) can be written that

$$\mathrm{v\,(x\,,t\,)} = \sum_{k=0}^{\infty} V_k^a(x)\,(t-t_o)^{ak},\ \mathrm{w\,(x\,,t\,)} = \sum_{k=0}^{\infty} W_k^a(x)\,(t-t_o)^{ak}. \quad (4.23)$$

Then u(x, t) is obtained as:

$$u(x,t) = v(x,t) = w(x,t)\ (\mathrm{x})(\,t-t_o)^{ak}$$

$$= \sum_0^{\infty} V_k^a(x)(t-t_o)^{ak} \sum_{k=0}^{\infty} W_o^a(x)(t-t_o)^{ak}$$

$$= [(V_0^a(x) + V_1^a x)(t-t_o\,)^a + V_2^a(x)(\,t-t_o)^{2a} + \cdots + V_n^a(x)(\,t-t_o)^{na} + \cdots]$$

$$\times\, [W_0^a(x) + W_1^a(x)(t-t_0)^a + W_2^a(x)(t-t_0)^{2a} + \cdots + W_n^a(x)(t-t_0)^{na} \ldots..]$$

$=[V_0^a(x)W_0^a(x) + (V_0^a(x)W_1^a(x) + V_1^a(x)W_0^a(x))(t-t_0)^a] +$

$(V_0^a(x)W_2^a(x) + \left(V_1^a(x)W_1^a(x)\right) + (V_2^a(x)W_0^a(x))(t-t_0)^{2a} + \cdots\ ]$

$$= \sum_{k=0}^{\infty}\sum_{s=0}^{k} V_s^{a}(x)\, W_{k-s}^{a}\,(\mathrm{x})(\mathrm{t}\,-t_o)^{ak}. \tag{4. 24}$$

Hence $U_k^a(x)\ is$ found as $U_k^a(x) = \sum_{s=0}^{k} V_s^{k}(x)W(x)$ .And the proof is completed.

Note; If $\mathrm{u(x,t)} = v_1(x\,,t\,)v_2(x\,,t)\ \ldots v_n(x\,,t)$,it is clear that

$$U_k^a(\mathrm{x}) = \sum_{k=0}^{\infty}\sum_{kn-2}^{kn-1}\ldots\sum_{k2=0}^{k_3}\sum_{k_{1=0}}^{k_2}(V_1)_{k_1}^{a}\,(x)(V_2)_{k_2-k1}^{a}\,(x)\ldots(V_n)_{k-kn-1}^{a}\,(x).$$

Theorem 4.6 (Keskin & Oturanc, 2010)

i) If $\mathrm{u(x\,,t\,)} = {}_tT_\alpha(\mathrm{v(x\,,t)})$ then , $U_k^a(\mathrm{x}) = \alpha(k+1)V_{k+1}^a(\mathrm{x})$ and

ii) If $\mathrm{u(x,t)} = {}_tT_{2\alpha}\left(\mathrm{v(x,t)}\right)$then, $U_k^a(\mathrm{x}) = a^2(k+2)(k+1)V_{k+2}^a(\mathrm{x})$.

Proof: - Here, we show the proof of (i) only the remains (ii) can be proved in similar way. Using definition 4.7, Let conformable fractional reduced differential transform of v(x, t) is as in the following.

$$V_k^a(\mathrm{x}) = \frac{1}{a^k k!}[{}_tT_\alpha v)^{(K)}]_{t-t_o}$$

For $V_k^a(x,t) = {}_tT_\alpha\big(v(x,t)\big),\ U_k^a(x) = \frac{1}{a^k k!}\,[{}_tT_\alpha v)^{(K)}]_{t=t_o}$ (4.25)

$$=\frac{1}{a^k k!}\,[({}_tT_\alpha v)^{K+1}]_{t=t_o}$$

$$= \alpha\,(k+1)\,\frac{1}{a^{k+1}(k+1)!}[{}_tT_\alpha v)^{K+1}]_{t=t_o}$$

$$= \alpha(k+1)V_{k+1}^a(x).\ \text{Proved} \tag{4.26}$$

Table 1:Basic properties of conformable fractional reduced differential transform method (Keskin & Oturanç, 2010).

Original function	Transformed function
$u(x,t).$	$U_k^{\alpha}(x,t) = \frac{1}{\alpha^k k!}[\left({}_tT_{\alpha}^{(k)}u\right)]_{t\,=\,t_0}$.
$u(x,t) = u_1(x,t)u_2(x,t)u_3(x,t)$	$U_k^{\alpha}(x,t) \;= \sum_{j=0}^{k}\sum_{i=0}^{j} U_i^{\alpha}(x,t)U_{j-i}^{\alpha}(x,t)U_{k-j}^{\alpha}(x,t)$.
$u(x,t) = v(x,t) \pm w(x,t).$	$U_k^{\alpha}(x,t) = V_k^{\alpha}(x,t) \,\pm\, W_k^{\alpha}(x,t).$
$u(x,t) = \; cv(x,t).$	$U_k^{\alpha}(x,t) = cV_k^{\alpha}(x,t).$
$u(x,t) = v(x,t)w(x,t).$	$U_k^{\alpha}(x,t) = \sum_{s=0}^{k} V_s^{\alpha}(x,t)W_{k-s}^{\alpha}\,(x,t).$
$u(x,t) = {}_tT_{\alpha}v(v(x,t)),$	$U_k^{\alpha}(x,t) = \alpha(k+1)V_{k+1}^{\alpha}(x,t).$

CHAPTER FIVE

RESULT AND DISCUSSION

5.1 conformable reduced differential transform method for fractional Allen-Cahn equation

Under this section, the reduced differential transform method procedures for solving one dimensional homogeneous time fractional Allen-Cahn equation of the following form is newly introduced and developed.

Consider one dimensional homogeneous time fractional Allen-Cahn equation of form:

$$\partial_t^\alpha u(x,t) - u_{xx}(x,t) + u^3(x,t) - u(x,t) = 0,\ \ 0 < \alpha \le 1, t > 0, \tag{5.1}$$

with initial condition $u(x,0) = h(x)$. (5.2)

Applying CRDTM on Equ.(5.1), we get

$$\alpha(k+1)U_{k+1}^\alpha(x) = \frac{\partial^2 U_k^\alpha(x)}{\partial x^2} + U_k^\alpha(x) - F_k(x)$$

$$U_{k+1}^\alpha(x) = \frac{1}{\alpha(k+1)}\left[\frac{\partial^2 U_k^\alpha(x)}{\partial x^2} + U_k^\alpha(x) - F_k(x)\right], \tag{5.3}$$

where $U_k^\alpha(x)$is the CFRDT of function $u(x,t)$and $F_k(x) = \sum_{j=0}^{k}\sum_{i=0}^{j} U_i^\alpha(x) U_{j-i}^\alpha(x)\, U_{k-j}^\alpha(x)$.

Applying the CFRDT to initial condition (5.2), in the sense of equation (4.12), we get

$$U_0^\alpha(x) = u(x,0) = h(x). \tag{5.4}$$

Substituting equation (5.4) into equation (5.3) and a straightforward calculation yields the following successive values of $U_k^\alpha(x)$:

For k = 0 , $U_1^\alpha(x) = \frac{1}{\alpha}\left(\frac{\partial^2 u}{\partial x^2} U_0^\alpha(x) + U_0^\alpha(x) - F_o(x)\right)$, and $=F_o(x) = \left(U_0^\alpha(x)\right)^3 = \left(h(x)\right)^3$.

. $$U_1^\alpha(x) = \frac{1}{\alpha}\left(\frac{\partial^2 u}{\partial x^2} h(x) + h(x) - (h(x))^3\right),$$

For $k = 1$,

$$U_2^\alpha(x) = \frac{1}{2\alpha}\left(\frac{\partial^2 u}{\partial x^2} U_1^\alpha(x) - U_1^\alpha(x) + F_1(x)\right),$$

For $k = 2$,

$$U_3^\alpha(x) = \frac{1}{3\alpha}\left(\frac{\partial^2 u}{\partial x^2} U_2^\alpha(x) - U_2^\alpha(x) + F_2(x)\right)$$ and so on.

Taking the inverse CFRDT of the set of values $(U_k^a(x))_{k=0}^{n}$ gives the n-terms approximate

Solutions as:

$$\tilde{u}_n(x) = \sum_{k=0}^{\infty} U_k^a \quad (x)t^{ak}$$

$$=U_0(x) + U_1(x)t^a + U_2(x)t^{2a} + \cdots \quad (5.5)$$

$$=U_o(x) + \frac{1}{a(k+1)}\left[\frac{\partial^2 u}{\partial x^2} U_0^a(x) + U_0^a(x) - U_0^3(x)\right] t^a + \cdots$$

Therefore, by sense of equation (4.16), the solution of the problem (5.1) is given by

$$\mathrm{u(x,t)} = \lim_{n\to\infty} \tilde{u}_n(x) = \sum_{k=0}^{\infty} U_k^a \,(\mathrm{x})t^{ak}. \quad (5.6)$$

After calculating each values of $U_k^a(x)t^{ak}$ and for α = 1, was becomes

$$u(x,t) = U_O(x,t) - U_1(x,t) + U_2(x,t) = 0{,}0 < a \leq 1, t > 0, \quad (5.7)$$

which converge to the exact solution of (5.1).

5.2 Convergence Analysis

In this section, we study the convergence analysis of the approximated analytical solutions of equation (5.1) by CFRDTM. This can be done as the same manner of convergence in application of RDTM(Al-Saif & Harfash, 2018),with conformable fractional. So, the sufficient condition that guarantees for existence of unique solution and convergence of the solution are discussed here.

Consider the Allen-Cahn equation (5.1) be given in the following functional equation form:

$$\mathrm{U}\,(x,t) = N(u\,(x,t)), \quad (5.8)$$

where N is the general linear and non-linear operator involving in both linear and non-linear terms. According to CFRDTM, the fractional Allen-Cahn equation given in equation (5.1) has a solution of the form:

$$u(x,t) = \sum_{k=0}^{\infty} U_k^a(x)t^{ka=\sum_{k=0}^{\infty} B_k^a,} \;\; 0 < a \leq 1\,, \quad (5.9)$$

where B_k^a is the transform of sequences function s (t).

It is noted that the solution by CFRDTM is equivalent to the determining sequences:

$s_0^a = U_0^a(x) = B_0^a\,, \forall_{k=0\,,1\,2\ldots..}$

$s_1^a = U_0^a(x) + U_1^a(x) = B_0^a + B_1^a,$

$s_2^a = U_0^a(x) + U_1^a(x) + U_2^a(x) = B_0^a + B_1^a\,, \ldots$

$$s_n^a = \sum_{k=0}^{n} U_k^a(x)t^{ka} = \sum_{k=0}^{n} B_k^a\,. \quad (5.10)$$

By the help of iterative scheme, we get $S_{n+1}^{a} = N(S_n^a)$ which is associated with the functional equation $\mathrm{S} = \mathrm{N}\ (\mathrm{s})$.

Hence, the solution $\mathrm{u}\ (\mathrm{x}, \mathrm{t}) = \sum_{k=0}^{n} U_k^a(x) t^{ka} = \sum_{k=0}^{n} B_k^a$ obtained by CFRDTM is equivalent to: $\mathrm{u}(\mathrm{x}, \mathrm{t}) = U_0^a(x) + U_1^a(x)t^a + U_2^a(x)t^{2a} + U_3^a(X)t^{3a} + \cdots = (s_n^a)_{n=0}^{\infty}$. (5.11)

The sufficient condition for convergence of the series solution $(s_n^a)_{n=0}^{\infty}$ is given in the following theorems.

Theorem 5.1 (Al-Saif & Harfash, 2018) Let N be an operator from a Hilbert space $\mathcal{H}$ into $\mathcal{H}$ then the series solution $(s_n^a)_{n=0}^{\infty}$ Converges,

whenever there is "δ "such that $0 < \delta \leq 1$, and $\| B_{k+1}^a \| \leq \delta \| B_k^a \|$.

Proof; Suppose that $\| s_{n+1}^a - s_n^a \| \leq \| B_{n+1}^a \| + \delta \| B_n^a \| + \delta^2 \| B_{n-1}^a \| \leq \delta^{n+1} \| B_0^a \|$.

We want to that $(s_n^a)_{n=0}^{\infty}$ is a Cauchy sequence in Hilbert space $\mathcal{H}$.

For every, $\mathrm{n}, \mathrm{m} \in \mathbb{N}$, $n \geq m$, by a formula for the sum of geometric sequences and triangular inequality, we find that

$$\begin{aligned}
\| s_n^a - s_m^a \| &= \| s_n^a - s_{n-1}^a + s_{n-2}^a + \ldots\ldots s_{m+1}^a - s_m^a \| \\
&\leq \| s_n^a - s_{n-1}^a \| + \| s_{n-1}^a - s_{n-2}^a \| + \cdots \| s_{m+1}^a - s_m^a \| \\
&\leq \delta^n \| B_O^a \| + \delta^{n-1} \| B_o^a \| + \delta^{n-2} \| B_o^a \| + \cdots \delta^{m+1} \| B_o^a \| \\
&= \delta^{m+1}(\delta^{n-m-1} + \delta^{n-m-2} + \cdots + 1) \| B_O^a \| \\
&= \delta^{m+1}\left(\frac{1-\delta^{n-m-1}}{1-\delta}\right).
\end{aligned}$$

Since $0 < \delta \leq 1$, the numerator $1 - \delta^{n-m-1} < 1$

Consequently , $\| s_n^a - s_m^a \| \leq \frac{\delta^{m+n}}{1-\delta} \| B_o^a \|$

On the right-hand side $0 < \delta \leq 1$ and $\| B_o^a \| < \infty$, then we have

$lim_{n,m \to \infty} \| s_n^a - s_m^a \| = 0$.

Thus, we conclude that $\{s_n^a\}_{n=0}^{\infty}$ is a Cauchy sequence in Hilbert space $\mathscr{H}$, hence the series $\{s_n^a\}_{n=0}^{\infty}$ Converges to some (S) $\epsilon\mathcal{H}$.

Theorem 5.2 (Al-Saif & Harfash, 2018) Let N be a non-linear operator satisfies the Lipschitz condition from Hilbert space $\mathscr{H}$ to $\mathscr{H}$ and $u(x, \text{t})$ be exact solution of the given Allen-Cahn equation (5.1). If the series $(s_n^a)_{n=0}^{\infty}$ converges, then it converges to u (x, t).

Proof: Under this theorem, we prove uniqueness of the solution and Convergence of the solution the one-dimensional Allen-Cahn equation given in equation (5.1).

I) Uniqueness of the solution

Let $u_1(x,\text{t})$ and $u_2(\text{x,t})$ be the solution of the given one-dimensional Allen-Cahn equation, then we show $u_1(x,t) = u_2(x,t)$. Since N satisfies the Lipschitz condition in Hilbert space and $0 < \delta \leq 1$, we have

$\| N(u_1) - N(u_2) \| \leq \delta \| u_1 - u_2 \|$ .Since $u_1 = \mathrm{N}(u_1)$ and $u_2 = \mathrm{N}(u_2)$,

we get $\| u_1 - u_2 \| = \| N(u_1) - N(u_2) \| \leq \delta \| u_1 - u_2 \|$.

$= \| u_1 - u_2 \| = 0$.

Since $0 < \delta$, $u_1 = u_2$. Thus, $u_1\,(\mathrm{x,t}) = u_2(x,t)$.

Therefore, from Banach fixed theorem, there is a unique solution of the problem (5.1).

Convergence of the solution

We show that the series $(s_n^a)_{n=0}^{\infty}$ converges to $\mathrm{u(x,t)}$;

Assume that, $\mathrm{u(x,t)} = \mathrm{N(x\,(x,t))}$ where N is the general non-linear operator.

Since S =N(S), we have

$$\mathrm{u(x,t)} = \mathrm{N}\big(\mathrm{u(x,t)}\big) = N(\textstyle\sum_{k=0}^{\infty} B_k^a) = N(lim_{n,\to\infty} \sum_{k=0}^{\infty} B_k^a)$$

$$= \lim_{n\to\infty} N(\textstyle\sum_{K=0}^{\infty} B_k^a) = \lim_{n\to\infty} N(S_n^a) = \lim_{n\to\infty} s_{n+1}^a = S$$

Since $s_n^a)_{n=0}^{\infty}$ converges to S and $\mathrm{S} = \mathrm{u\,(x,t)}$ then, the series solution $(s_n^a)_{n=0}^{\infty}$ converges to the exact solution. For particular α = 1, as $\mathrm{n} \to \infty, s_n^a \to \mathrm{u\,(x,t)}$.

Definition 5.1 (Al-Saif & Harfash, 2018) For $k \in \mathbb{N}\, U\, (0)$, we define

$$\delta_k^a = \begin{cases} \frac{\|B_{k+1}^a\|}{\|B_K^a\|} = \frac{\|U_{k+1}^a(x)t^{a(k+1)}\|}{\|U_k^a(x)t^{ak}\|}, if \ \| B_k^a \| = \| U_k^a(x)t^{ak} \| \neq 0, \\ 0, if \ \| B_k^a \| = \| U_k^a(x)t^{ak} \| = 0. \end{cases} \tag{5.12}$$

Then we can say that the series approximate solution $(S_n^a)_{n=0}^{\infty}$ converges to the solution u(x,t) when , $0 \leq \delta_k^a < 1$ for k = 0,1,2,. .

5.3 Illustrative Examples

To check the accuracy and efficiency of the CRDTM, we consider two testing examples in this section.

Example 1. Consider the time fractional Allen-Cahn equation stated as:

$$D_t^{\alpha} u\,(x,t) \;=\; D_{xx} u\,(x,t) \;+\; u\,(x,t) \;-\; u^3(x,t), \quad 0<\alpha\le 1, t>0, \tag{5.13}$$

subject to initial condition

$$u\,(x,0) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)} \;. \tag{5.14}$$

Apply the CFRDTM to Eq. (5.13) and (5.14) to get

$$U_{k+1}^{\alpha}(x) = \frac{1}{\alpha(k+1)}\left(\frac{\partial^2}{\partial t^2} U_k^{\alpha}(x) + U_k^{\alpha}(x) - \sum_{j=0}^{k}\sum_{i=0}^{j} U_i^{\alpha}(x) U_{j-i}^{\alpha}(x) U_{k-j}^{\alpha}(x)\right), \tag{5.15}$$

and

$$U_0^{\alpha}(x) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)}. \tag{5.16}$$

where the function $U_k^{\alpha}(x)$ is the conformable time fractional reduced differential transform function.

Substituting Eq. (5.16) in to Eq. (5.15), respectively, we obtain:

$$U_1^{\alpha}(x) = \frac{3e^{\frac{x}{\sqrt{2}}}}{2(1+e^{\frac{x}{\sqrt{2}}})^2\alpha}$$

$$U_2^{\alpha}(x) = \frac{9e^{\frac{x}{\sqrt{2}}}\left(-1+e^{\frac{x}{\sqrt{2}}}\right)}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3\alpha^2}$$

$$U_3^{\alpha}(x) = \frac{9e^{\frac{x}{\sqrt{2}}}\left(1-4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x}\right)}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4\alpha^3}$$

$$U_4^{\alpha}(x) = \frac{27e^{\frac{x}{\sqrt{2}}}\left(-1+11e^{\frac{x}{\sqrt{2}}}+e^{\frac{3x}{\sqrt{2}}}-11e^{\sqrt{2}x}\right)}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5\alpha^4}$$

And, so on. Thus by using the inverse CFRDTM. i.e. $u(x,t) = \sum_{k=0}^{\infty} U_k^{\alpha}(x) t^{k\alpha}$

we get the approximated solution:

$u(x,t) =$

$$\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)}+\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2\alpha}t^{\alpha}+\frac{9e^{\frac{x}{\sqrt{2}}}\left(-1+e^{\frac{x}{\sqrt{2}}}\right)}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3\alpha^2}t^{2\alpha}+\frac{9e^{\frac{x}{\sqrt{2}}}\left(1-4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x}\right)}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4\alpha^3}t^{3\alpha}+\frac{27e^{\frac{x}{\sqrt{2}}}\left(-1+11e^{\frac{x}{\sqrt{2}}}+e^{\frac{3x}{\sqrt{2}}}-11e^{\sqrt{2}x}\right)}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5\alpha^4}t^{4\alpha}+\ldots \quad (5.17)$$

In particular when $\alpha = 1$, we get the solution in the form

$$u(x,t)=\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)}+\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}t+\frac{9e^{\frac{x}{\sqrt{2}}}\left(-1+e^{\frac{x}{\sqrt{2}}}\right)}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}t^2+\frac{9e^{\frac{x}{\sqrt{2}}}\left(1-4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x}\right)}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}t^3+\frac{27e^{\frac{x}{\sqrt{2}}}\left(-1+11e^{\frac{x}{\sqrt{2}}}+e^{\frac{3x}{\sqrt{2}}}-11e^{\sqrt{2}x}\right)}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}t^4+\ldots \quad (5.18)$$

and which converge rapidly to the classical solution

$u(x,t) = \frac{1}{1+e^{-\frac{3t}{2}+\frac{x}{\sqrt{2}}}}$ (Rawashdeh, 2017, Ali et al., 2022).

Example 2. Consider the time fractional Allen-Cahn equation stated as:

$$D_t^{\alpha}u\,(x,t) = D_{xx}u\,(x,t) + u\,(x,t) - u^3(x,t), \quad 0 < \alpha \le 1, \quad t > 0, \qquad (5.19)$$

subject to initial condition

$$u\,(x,0) = \frac{1}{\left(1+e^{-\frac{x}{\sqrt{2}}}\right)} \quad . \qquad (5.20)$$

Applying the Conformable reduced differential transform method to (5.19 and 5.20), we obtain

$$U_{k+1}^{\alpha}(x) = \frac{1}{\alpha(k+1)}\left(\frac{\partial^2}{\partial t^2}U_k^{\alpha}(x) + U_k^{\alpha}(x) - \sum_{j=0}^{k}\sum_{i=0}^{j}U_i^{\alpha}(x)U_{j-i}^{\alpha}(x)U_{k-j}^{\alpha}(x)\right), \qquad (5.21)$$

And $U_0^{\alpha}(x) = \frac{1}{\left(1+e^{-\frac{x}{\sqrt{2}}}\right)}$. (5.22)

Substituting Eq. (5.22) in to Eq. (5.21), respectively, we get:

$$U_1^{\alpha}(x)=\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2\alpha}$$

$$U_2^{\alpha}(x)=-\frac{9e^{\frac{x}{\sqrt{2}}}\left(-1+e^{\frac{x}{\sqrt{2}}}\right)}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3\alpha^2}$$

$$U_3^{\alpha}(x)=\frac{9e^{\frac{x}{\sqrt{2}}}\left(1-4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x}\right)}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4\alpha^3}$$

$$U_4^{\alpha}(x)=-\frac{\left(27e^{\frac{x}{\sqrt{2}}}\left(-1+11e^{\frac{x}{\sqrt{2}}}+e^{\frac{3x}{\sqrt{2}}}-11e^{\sqrt{2}x}\right)\right)}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5\alpha^4}$$

$$U_5^{\alpha}(x)=\frac{81e^{\frac{x}{\sqrt{2}}}\left(1-26e^{\frac{x}{\sqrt{2}}}-26e^{\frac{3x}{\sqrt{2}}}+66e^{\sqrt{2}x}+e^{2\sqrt{2}x}\right)}{1280\left(1+e^{\frac{x}{\sqrt{2}}}\right)^6\alpha^5}$$

and so on.

Thus, using the inverse CFRDTM, i.e. $u(x,t)=\sum_{k=0}^{\infty}U_k^{\alpha}(x)t^{k\alpha}$,

we get the approximated solution:

$$u(x,t)=\frac{1}{\left(1+e^{-\frac{x}{\sqrt{2}}}\right)}+\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2\alpha}t^{\alpha}-\frac{9e^{\frac{x}{\sqrt{2}}}\left(-1+e^{\frac{x}{\sqrt{2}}}\right)}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3\alpha^2}t^{2\alpha}+\frac{9e^{\frac{x}{\sqrt{2}}}\left(1-4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x}\right)}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4\alpha^3}t^{3\alpha}-\frac{27e^{\frac{x}{\sqrt{2}}}\left(-1+11e^{\frac{x}{\sqrt{2}}}+e^{\frac{3x}{\sqrt{2}}}-11e^{\sqrt{2}x}\right)}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5\alpha^4}t^{4\alpha}+\ldots \quad (5.23)$$

In particular when $\alpha=1$, we get the solution in the form:

$$u(x,t)=\frac{1}{\left(1+e^{-\frac{x}{\sqrt{2}}}\right)}+\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}t-\frac{9e^{\frac{x}{\sqrt{2}}}\left(-1+e^{\frac{x}{\sqrt{2}}}\right)}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}t^2+\frac{9e^{\frac{x}{\sqrt{2}}}\left(1-4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x}\right)}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}t^3-\frac{27e^{\frac{x}{\sqrt{2}}}\left(-1+11e^{\frac{x}{\sqrt{2}}}+e^{\frac{3x}{\sqrt{2}}}-11e^{\sqrt{2}x}\right)}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}t^4$$

$$+\frac{81e^{\frac{x}{\sqrt{2}}}\left(1-26e^{\frac{x}{\sqrt{2}}}-26e^{\frac{3x}{\sqrt{2}}}+66e^{\sqrt{2}x}+e^{2\sqrt{2}x}\right)}{1280\left(1+e^{\frac{x}{\sqrt{2}}}\right)^6}t^5+\ldots, \quad (5.24)$$

and converge rapidly to the classical solution

$$u(x,t)=\frac{1}{1+e^{-\frac{3t}{2}-\frac{x}{\sqrt{2}}}}$$ (Esen et al. 2013).

5.4. Numerical results and discussion.

In this section, the efficiency and accuracy of the Conformable reduce differential transform method were discussed and presented in the form of tables and graphs based on the numerical results of the Examples 1 and 2 considered in the above section. The numerical results were generated by using Mathematical software.

Table 2: Numerical values of Example 1 derived from CFRDTM for x=1 and different fractional order.

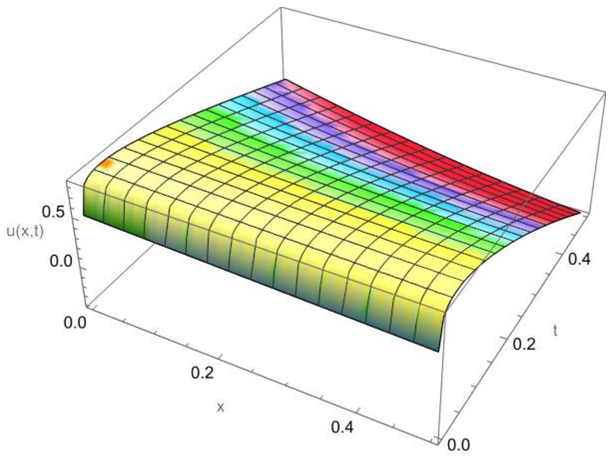
T	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
0.2	0.54701206	0.64868204	0.49673278	0.42175044
0.4	-0.02597964	0.73563379	0.60189638	0.50570834
0.6	-0.73425283	0.74474292	0.67934849	0.58419471
0.8	-1.52235250	0.69280893	0.72912901	0.65315441
1.0	-2.36729134	0.58691158	0.74883637	0.70741405

Table 3: Absolute errors of Example 1 obtained by CFRDTM in comparison with NITM and HPM, when α = 1 and x=1 and for diverse values of t.

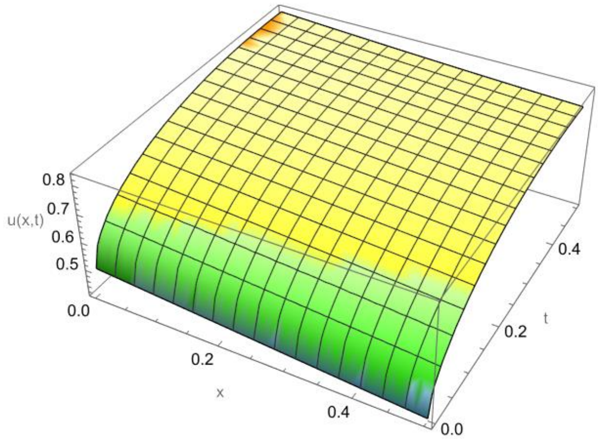
T	u_{App} (by CFRDTM)	u_{Exact}	Absolute Errors		
			CFRDTM	HPM	NITM
0.001	0.330570	0.330570	5.553×10^{-17}	7.6×10^{-12}	4.3×10^{-12}
0.002	0.330902	0.330902	5.551×10^{-17}	3.8×10^{-10}	3.5×10^{-11}
0.003	0.331235	0.331235	7.772×10^{-16}	1.1×10^{-09}	1.1×10^{-10}
0.004	0.331567	0.331567	3.386×10^{-15}	2.6×10^{-09}	2.8×10^{-10}
0.005	0.331899	0.331899	1.033×10^{-14}	5.0×10^{-09}	5.5×10^{-10}
0.006	0.332232	0.332232	2.592×10^{-14}	9.7×10^{-09}	9.6×10^{-10}
0.007	0.332565	0.332565	5.601×10^{-14}	1.5×10^{-08}	1.5×10^{-09}
0.008	0.332898	0.332898	1.095×10^{-13}	2.0×10^{-08}	2.2×10^{-09}
0.009	0.333231	0.333231	1.977×10^{-13}	2.9×10^{-08}	2.3×10^{-09}
0.01	0.333565	0.333565	3.354×10^{-13}	4.0×10^{-08}	4.5×10^{-09}

Table 2 shows the numerical values of the one dimensional time fractional Allen-Cahn equation given in Example 1 for different fractional order $\alpha \in [0.3, 0.5, 0.7, 0.9]$. **Table 3** show comparisons of the absolute errors obtained from CFRDTM and the results obtained by NITM and HPM (Ali et al. 2022). It can be observed that the absolute errors obtained from CFRDTM match better than results obtained by NITM and HPM, and also better approximations can be achieved when t is small.

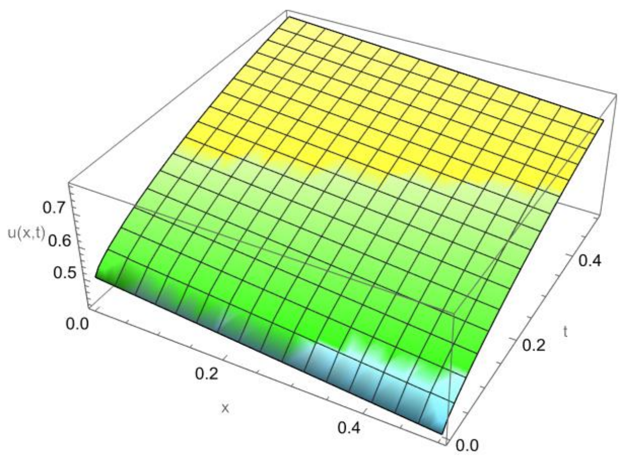
In **Figure 1**, the graphical depiction of the behaviour of the approximate solution of Example 1 for $\alpha = 0.3$, $\alpha = 0.5$, $\alpha = 0.7$ and $\alpha = 0.9$ is given, and in Figure 2, the graphs of the approximate solution by FRDTM and the exact solution for $\alpha = 1$ were depicted. It can be observed from the figure that there is a good agreement between exact solution and approximate solution at $\alpha = 1$.



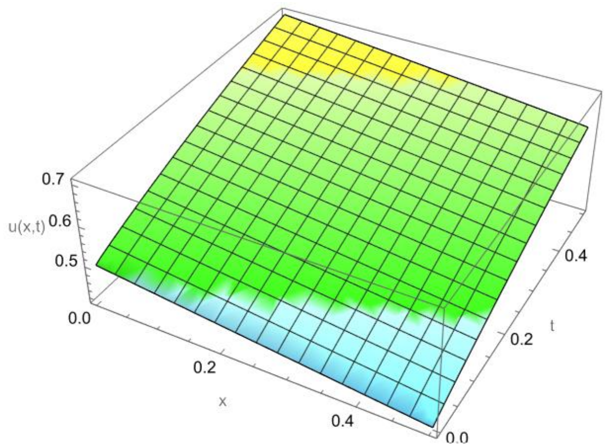
A



B

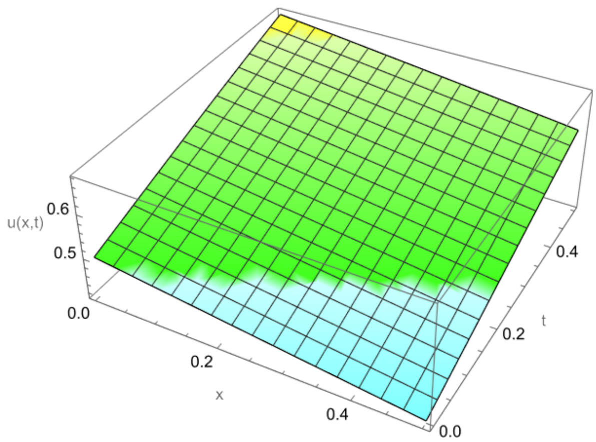


C

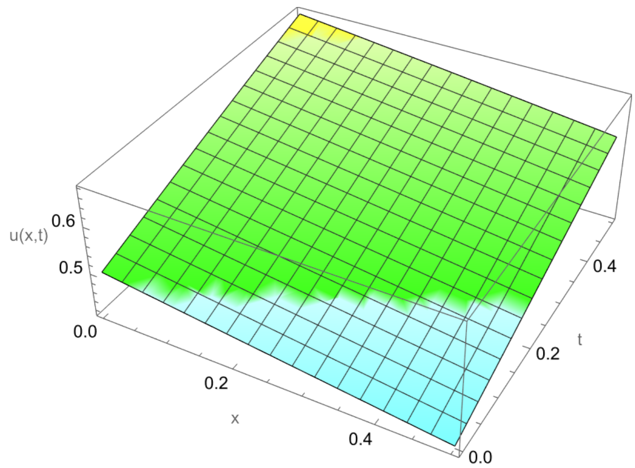


D

Figure 1: Solution behavior of Example 1 when a) α = 0.3, b) α = 0.5, c) α = 0.7, d) α = 0.9.



A



B

Figure 2: Solutions behavior of Example 1: a) Approximate b) Exact.

In Table 4, numerical values of the time fractional Allen-Cahn equation given in Example 2 obtained by CFRDTM for various values of the fractional order α were presented.

Table 5**,** show the comparison of absolute errors between HAM (Esen et al. 2013) and CFRDTM. From this table we can see that results are in a good agreement. However, the CFRDTM results are better that HAM for small values of t .

In Figures 3 and 4 the approximate solutions for several values of $\alpha \in [0.3, 0.5, 0.7, 0.9, 1]$ and exact solutions of Example 2 were depicted. From these figures we see that the approximate solutions are closer and closer to the exact solution when the value of the fractional order α approaches to 1.

Table 4: Numerical values of Example 2 obtained by CFRDTM by fifth approximation for x=1and different fractional order α.

T	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
0.2	1.23071575	0.88845572	0.80250702	0.75000834
0.4	1.72731241	0.94962592	0.86338408	0.80818293
0.6	2.35772302	1.00996884	0.90570315	0.85380196
0.8	3.09395448	1.08855408	0.94216419	0.89089324
1.0	3.91884302	1.19436426	0.98137664	0.92378465

Table 5: Absolute errors of Example 2 obtained by CFRDTM by fifth approximation compared with results by HAM for α = 1 and for diverse values of x and t.

T	x	CFRDTM	HAM (Esen et al. 2013)	Exact	Absolute Errors	
					CFRDTM	HAM
0.5	0.2	0.70924197	0.710231	0.70918340	5.86×10^{-05}	1.05×10^{-03}
	0.4	0.73754922	0.738428	0.73746664	8.26×10^{-05}	9.61×10^{-04}
	0.6	0.76401092	0.764769	0.76391490	9.60×10^{-05}	8.54×10^{-04}
	0.8	0.78856124	0.789202	0.78846298	9.83×10^{-05}	7.39×10^{-04}
	1.0	0.81118029	0.811714	0.81108977	9.05×10^{-05}	6.24×10^{-04}
0.75	0.2	0.78088166	0.782775	0.78012953	7.52×10^{-04}	2.65×10^{-03}
	0.4	0.80440523	0.805400	0.80342546	9.79×10^{-04}	1.97×10^{-03}
	0.6	0.82589473	0.826175	0.82480742	1.09×10-03	1.37×10-03
	0.8	0.84538877	0.845168	0.84431428	1.07×10-03	8.54×10-04
	1.0	0.86297226	0.862458	0.86201251	9.59×10-04	4.46×10-04

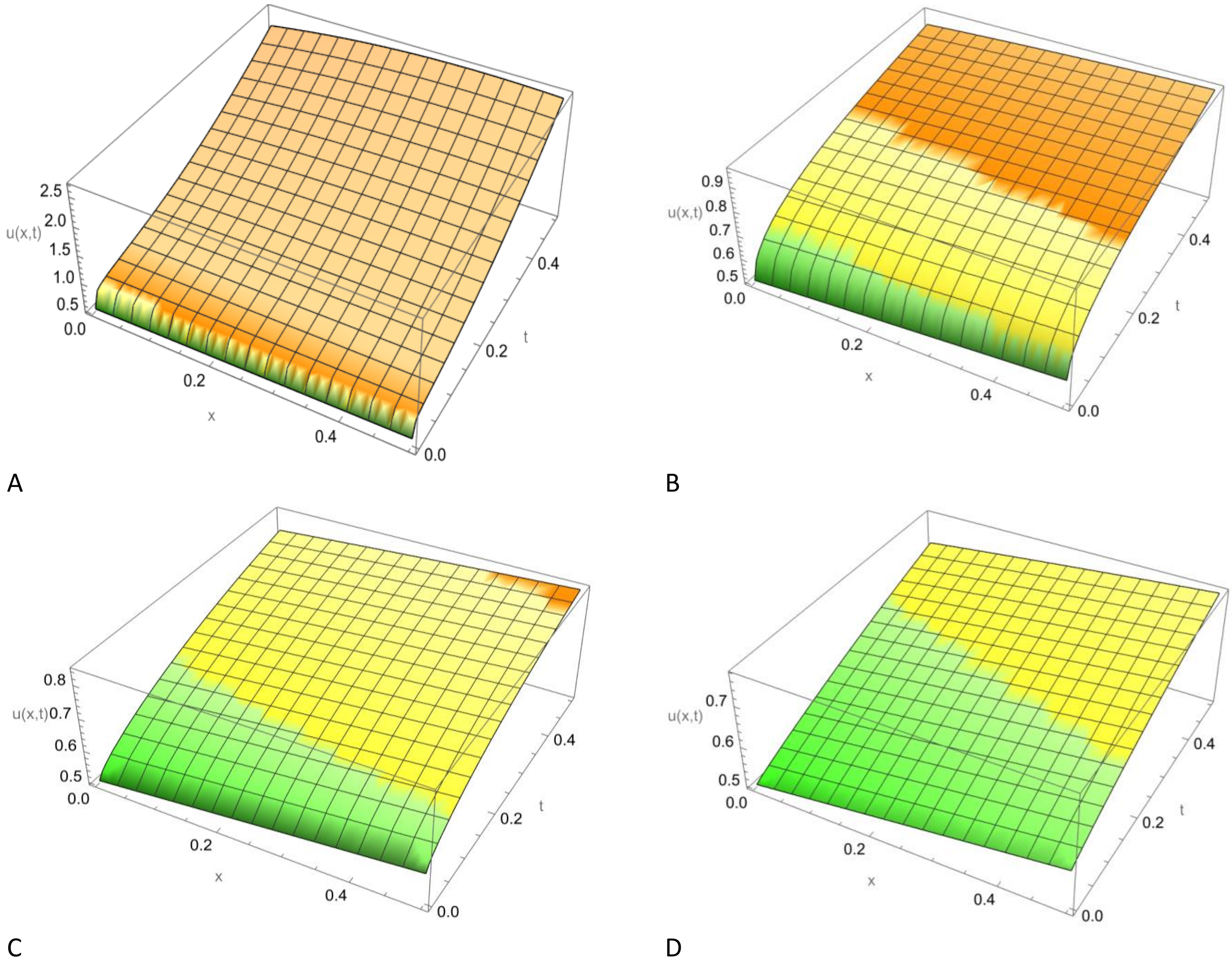
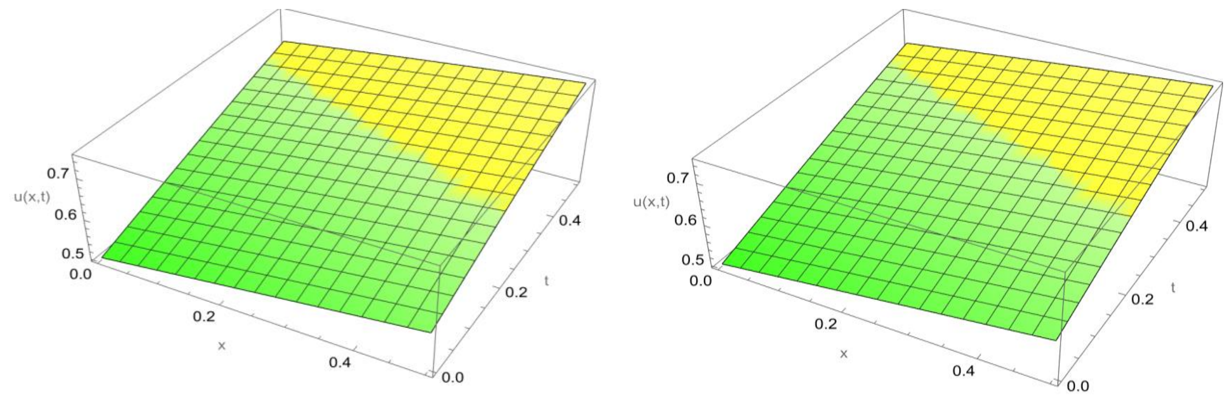


Figure 3: Solutions behavior of Example 2 when a) α = 0.3, b) α = 0.5, c) α = 0.7 and d) α = 0.9.



A **B**

Figure 4: Solutions behavior of Example 2: a) Approximate b) Exact.

CHAPTER SIX
CONCLUSION AND FUTURE SCOPE

6.1 Conclusion

In this study, we have applied Conformable Reduced differential transform Method to obtain an approximate analytical solution of the time fractional Allen-Cahn equation. To show the accuracy and efficiency of the proposed method two examples were considered. The behaviors of the approximate solution of the time fractional Allen-Cahn equation were presented in tabular and graphical forms.

Numerical results of both Examples obtained by CRDTM were compared with the classical solutions and with results in the exiting literature for $\alpha = 1$, and show that our result is in an excellent agreement with the results of Ali et al. 2022 and Esen et al. 2013 respectively for Example 1 and 2 as shown in Tables 1 and 2. Consequently, the presented method is the reliable techniques to handle nonlinear time fractional Allen-Cahn equation. Generally, it can be noticed that solution depends on the time-fractional derivative. Accuracy and efficiency can be enhanced by increasing the number of iterations.

6.2 Future Scope

There are many areas that may be of future research that we encountered throughout the study. Following the same procedure, as discussed in this research work, the conformable reduced differential transform method may be applied to two or more-dimensional nonlinear Equations.

Also, the current study is limited on obtaining approximate analytical solutions with initial condition. The CFRDTM may eventually be extended to include boundary conditions. By enabling the suggested method to address more difficult real-world issues with precise boundary constraints, this analysis would increase the method's applicability.

REFERENCES

Acan, O., Firat, O. F., Keskin, Y., & Oturanç, G. (2016). Solution of Conformable Fractional Partial Differential Equations by Reduced Differential Transform Method. *Selcuk Journal of Applied Mathematics*. https://api.semanticscholar.org/CorpusID:123753258

Ahmad, H., Seadawy, A. R., Khan, T. A., & Thounthong, P. (2020). Analytic approximate solutions for some nonlinear Parabolic dynamical wave equations. *Journal of* Taibah *University for Science*, *14*(1), 346–358. https://doi.org/10.1080/16583655.2020.1741943

Al-Saif, A.-S. J., & Harfash, A. J. (2018). A Comparison between the Reduced Differential Transform Method and Perturbation-Iteration Algorithm for Solving Two-Dimensional Unsteady Incompressible Navier-Stokes Equations. *Journal of Applied Mathematics and Physics*, *06*(12), 2518–2543. https://doi.org/10.4236/jamp.2018.612211

Ali, L., Shah, R., & Weera, W. (2022). Fractional View Analysis of Cahn–Allen Equations by New Iterative Transform Method. *Fractal and Fractional*, *6*(6), 1–12. https://doi.org/10.3390/fractalfract6060293

Beyer, H. R., & Kempfle, S. (1995). Definition of physically consistent damping laws with fractional derivatives. *Zamm-Zeitschrift Fur Angewandte Mathematik Und Mechanik, 75*, 623–635. https://api.semanticscholar.org/CorpusID:122813919

Deresse, A. T. (2022). *Analytical Solution of One-Dimensional Nonlinear Conformable Fractional Telegraph Equation by Reduced Differential Transform Method*. *2022*.

Esen, A., Yagmurlu, N. M., & Tasbozan, O. (2013). *Approximate Analytical Solution to Time-Fractional Damped Burger and Cahn-Allen Equations*. *1956*(5), 1951–1956.

Gubes, M. (n.d.). *Reduced Differential Transform Method for Solving Foam Drainage Equation(FDE)*.

Hilfer, R. (2000). *Applications Of Fractional Calculus In Physics*.

Hussain, S., Shah, A., Ayub, S., & Ullah, A. (2019). An approximate analytical solution of the Allen-Cahn equation using homotopy perturbation method and homotopy analysis method. *Heliyon*, *5*(12). https://doi.org/10.1016/j.heliyon.2019.e03060

İbiş, B., & Bayram, M. (2012). Approximate solutions some nonlinear evolutions equations by using the reduced differential transform method. *International Journal of Applied Mathematical Research*, *1*(3), 1–302. https://doi.org/10.14419/ijamr.v1i3.112

Jefain, O., & Algahtani, J. (2016). Comparing the Atangana – Baleanu and Caputo – Fabrizio derivative with fractional order : Allen Cahn model. *Chaos, Solitons and Fractals*, *0*, 1–8. https://doi.org/10.1016/j.chaos.2016.03.026

Jneid, M., & Chaouk, A. (2020). The conformable reduced differential transform method for solving newell-whitehead-segel equation with non-integer order. *Journal of Analysis and Applications*, *18*(1), 35–51.

Karunakar, P., & Chakraverty, S. (2019). Shifted Chebyshev polynomials based solution of partial differential equations. *SN Applied Sciences*, *1*(4), 1–9. https://doi.org/10.1007/s42452-019-0292-z

Kempfle, S., Schäfer, I., & Beyer, H. (2002). Fractional calculus via functional calculus: Theory and applications. *Nonlinear Dynamics*, *29*(1–4), 99–127. https://doi.org/10.1023/A:1016595107471

Keskin, Y., & Oturanç, G. (2010). Reduced differential transform method for generalized KdV equations. *Mathematical and Computational Applications*, *15*(3), 382–393. https://doi.org/10.3390/mca15030382

Khalid, N., Abbas, M., Iqbal, M. K., & Baleanu, D. (2020). A numerical investigation of Caputo time fractional Allen–Cahn equation using redefined cubic B-spline functions. *Advances in Difference Equations*, *2020*(1). https://doi.org/10.1186/s13662-020-02616-x

Khalil, R., Al Horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, *264*, 65–70. https://doi.org/10.1016/j.cam.2014.01.002

Kilbas, A. A. A., Srivastava, H., & Trujillo, J. (2006). Theory and applications Of Fractinal Differential Equations. In *North Holland Mathematical Studies* (Vol. 204). https://doi.org/10.1016/S0304-0208(06)80001-0

Kim, Y., Ryu, G., & Choi, Y. (2021). Fast and Accurate Numerical Solution of Allen-Cahn Equation. *Mathematical Problems in Engineering*, *2021*. https://doi.org/10.1155/2021/5263989

Machado, J. T., Kiryakova, V., & Mainardi, F. (2011). Recent history of fractional calculus. *Communications in Nonlinear Science and Numerical Simulation*, *16*(3), 1140–1153. https://doi.org/https://doi.org/10.1016/j.cnsns.2010.05.027

Mainardi, F. (1996). Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos, Solitons & Fractals*, *7*(9), 1461–1477. https://doi.org/https://doi.org/10.1016/0960-0779(95)00125-5

MekonnenYadeta, & Obsie, Y. (2020). Approximate Analytical Solution of One-Dimensional Beam Equations by Using Time-Fractional Reduced Differential Transform Method. *Journal of Applied Mathematics*, *2020*.

https://doi.org/10.1155/2020/7627385

Miller, M., JK, R., Wlodawer, A., & MR, G. (1993). A left-handed crossover involved in amidohydrolase catalysis. Crystal structure of Erwinia chrysanthemi L-asparaginase with bound L-aspartate. *FEBS Letters*, *328*(3), 275–279.

Murat Gubes Richard B., 2010. (2018). *I nternational J ournal of M ultidisciplinary E ducational R esearch*. *7*(1), 1–272.

Odibat, Z. M., & Momani, S. (2008). A generalized differential transform method for linear partial differential equations of fractional order. *Appl. Math. Lett.*, *21*, 194–199. https://api.semanticscholar.org/CorpusID:10120125

Oldham, K. B., & Spanier, J. (1974). The Fractional Calculus, Acad. *Press, New York, London*, 130.

Petras, I. (2008). *Stability of Fractional-Order Systems with Rational Orders*. 1–25. http://arxiv.org/abs/0811.4102

Podlubny, I. (Ed.). (1999). Chapter 2 - Fractional Derivatives and Integrals. In *Fractional Differential Equations* (Vol. 198, pp. 41–119). Elsevier. https://doi.org/https://doi.org/10.1016/S0076-5392(99)80021-6

Prakash, A., Goyal, M., & Gupta, S. (2019). *Fractional variational iteration method for solving time-fractional Newell-Whitehead-Segel equation*. 164–171.

Ravi Kanth, A. S. V., Aruna, K., Raghavendar, K., Rezazadeh, H., & Inc, M. (2021). Numerical solutions of nonlinear time fractional Klein-Gordon equation via natural transform decomposition method and iterative Shehu transform method. *Journal of Ocean Engineering and Science*, *xxxx*. https://doi.org/10.1016/j.joes.2021.12.002

Rawashdeh, M. S. (2017). Open Access A reliable method for the space-time fractional Burgers and time-fractional Cahn-Allen equations via the FRDTM. *Adv Differ Equ*. https://doi.org/10.1186/s13662-017-1148-8

Rawashden(2013), M. (2013). *Mahmoud s. rawashdeh*. *6*(1), 11–18.

Ray, S. S., & Bera, R. K. (2005). Analytical solution of the Bagley Torvik equation by Adomian decomposition method. *Applied Mathematics and Computation*, *168*(1), 398–410. https://doi.org/https://doi.org/10.1016/j.amc.2004.09.006

Reza, & Malek. (2013). Numerical study of Burgers-Huxley equations via reduced differential transform method. *Computational and Applied Mathematics*, *32*(1), 1–17. https://doi.org/10.1007/s40314-013-0001-2

Safari, M., Ganji, D. D., & Moslemi, M. (2009). Application of He's variational iteration method and Adomian's decomposition method to the fractional KdV–Burgers–

Kuramoto equation. *Computers & Mathematics with Applications*, *58*(11), 2091–2097. https://doi.org/https://doi.org/10.1016/j.camwa.2009.03.043

Saravanan, A., & Magesh, N. (2013). A comparison between the reduced differential transform method and the Adomian decomposition method for the Newell–Whitehead–Segel equation. *Journal of the Egyptian Mathematical Society*, *21*(3), 259–265. https://doi.org/10.1016/j.joems.2013.03.004

Schneider, W. R., & Wyss, W. (1989). Fractional diffusion and wave equations. *Journal of Mathematical Physics*, *30*(1), 134–144. https://doi.org/10.1063/1.528578

Singh, B. K., & Kumar, P. (2017). *Fractional Variational Iteration Method for Solving Fractional Partial Differential Equations with Proportional Delay*. *2017*.

Sohail, M., & Mohyud-din, S. T. (2012). *Reduced Differential Transform Method for Time-fractional Heat Equations*. *1*(1), 13–22.

Srivastava, N., Hinton, G., Krizhevsky, A., Sutskever, I., & Salakhutdinov, R. (2014). Dropout: A simple way to prevent neural networks from overfitting. *Journal of Machine Learning Research*, *15*, 1929–1958.

Srivastava, V. K. (2015). Reduced differential transform method to solve two and three dimensional second order hyperbolic telegraph equations. *JOURNAL OF KING SAUD UNIVERSITY - ENGINEERING SCIENCES*. https://doi.org/10.1016/j.jksues.2014.04.010

Tamsir, M., & Srivastava, V. K. (2016). Revisiting the approximate analytical solution of fractional-order gas dynamics equation. *Alexandria Engineering Journal*, *55*(2), 867–874. https://doi.org/10.1016/j.aej.2016.02.009

Tarasov, V. E. (2010). Fractional Integration and Fractals. *Nonlinear Physical Science*, 3–48. https://doi.org/10.1007/978-3-642-14003-7_1

Thabet, H., & Kendre, S. (2018). Chaos , Solitons and Fractals Analytical solutions for conformable space-time fractional partial differential equations via fractional differential transform. *Chaos, Solitons and Fractals: The Interdisciplinary Journal of Nonlinear Science, and Nonequilibrium and Complex Phenomena*, *109*, 238–245. https://doi.org/10.1016/j.chaos.2018.03.001

Yang, X. J., Baleanu, D., Khan, Y., & Mohyud-Din, S. T. (2014). Local fractional variational iteration method for diffusion and wave equations on Cantor sets. *Romanian Journal of Physics*, *59*(1–2), 36–48.

Yang, Z., Su, W., Wei, X., Xiang, J., Yang, Z., Su, W., Wei, X., Qu, S., Zhao, D., Zhou, J., Wang, Y., & Guan, Q. (2023). Article HIF-1 a drives resistance to ferroptosis in solid tumors by promoting lactate production and activating SLC1A1 ll HIF-1 a drives resistance to ferroptosis in solid tumors by promoting lactate production and activating SLC1A1. *CellReports*, *42*(8), 112945. https://doi.org/10.1016/j.celrep.2023.112945

Zada, L., Ali, N., Nawaz, R., Jamshed, W., Eid, M. R., Sayed, E., El, M. T., Abd, H., & Khalifa, E.-W. (2022). *Applying the natural transform iterative technique for fractional high-dimension equations of acoustic waves*. *November*, 1–10. https://doi.org/10.3389/fenrg.2022.979773

Zhang, X., Zhao, J., Liu, J., & Tang, B. (2014). Homotopy perturbation method for two dimensional time-fractional wave equation. *Applied Mathematical Modelling*, *38*(23), 5545–5552. https://doi.org/https://doi.org/10.1016/j.apm.2014.04.018