A ROBUST NUMERICAL METHOD FOR A SINGULARLY PERTURBED TWO-PARAMETER DELAY DIFFERENTIAL EQUATION

JIMMA UNIVERSITY

COLLEGE OF NATURAL SCIENCE

DEPARTMENT OF MATHEMATICS

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(Numerical Analysis)

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Jimma, Ethiopia

DECLARATION

I undersigned this thesis entitled "A Robust Numerical Method for Singularly Perturbed Two-Parameter Delay Differential Equations" is my original work, and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that All the sources I have used or quoted have been indicated and acknowledged as complete references.

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This is to certify that the thesis entitled 'A ROBUST NUMERICAL METHOD FOR A SINGULARLY PERTURBED TWO-PARAMETER DELAY DIFFERENTIAL EQUATION" has been developed by **Demirew Dula Hundie** ,under my supervision. Therefore we recommend that the student's thesis be presented for review and open oral presentation.

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Abbreviations

AMR-Adaptive mesh refinement

BVPs-Boundary Value Problems

DDEs -Delay Differential Equations

DEs-Differential Equations

FDM-Finite Difference Methods

FMMs-Fitted Mesh Methods

FOMs-Fitted Operator Methods

FOFDM -Fitted Operator Finite Difference Method.

MATLAB-Matrix Laboratory

ODEs- Ordinary Differential Equations

RKM-Runge-Kutta Methods

SP-DDEs -Singularly Perturbed Delay Differential Equations

SP- Singular perturbed

SPPs-Singularly Perturbed Problems

Abstract

The aim of this thesis is to present an exponentially fitted numerical method for solving singularly perturbed differential equations with two parameters and a large delay. The accuracy and parameter uniform convergence of the proposed method is presented.To validate the applicability of the scheme,one model problem is considered for numerical experimentation and solved for different values of the perturbation parameter and mesh size. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence, and it was observed that the present method is more accurate and uniformly convergent independent of the parameters.

Chapter 1

INTRODUCTION

1.1 Background of the Study

Numerical analysis specifically focuses on the solution of the singularly perturbed delay differential equations (SPDDEs), which exhibit a multi-scale nature due to the presence of small parameters and the inclusion of delay terms. These types of problems arise in a wide range of applications, such as in the modeling of biological systems, network dynamics, and material sciences Debela and Duressa (2023). The Development of Numerical Algorithms are for finding numerical solutions to problems that cannot be easily solved using the analytical method. The theoretical foundations and experimental problem-solving focus on the development of a fitted operator method for singularly perturbed problems two-parameter delay differential equation. Singularly perturbed delay differential equation problems play an important role in the natural sciences, social sciences, engineering, medicine, and business applications where there are significant differences in the scales of the independent variables or parameters involved Amiraliyev and Duru (1999), Derstine (1984),Kadalbajoo and Sharma (2003),Miller *et al.*(1996) and O'Malley (1991) . Delay differential Equations, on the other hand, capture dynamic systems in which the derivative of the dependent variable depends not only on its current value but also on its past values. If the differential equation in which the highest-order derivative term is multiplied by the small positive parameter ε where, $0 < \varepsilon \ll 1$ is known to be a Singularly Perturbed differential equation and the parameter is known as the perturbation parameter Debela and Duressa (2022),Kadalbajoo and Sharma (2003) and File *et al.* (2017). and also the small parameter multiplied with other than the highest order derivative term is called Regular Perturbation Problems. There are two categories of perturbation problems: regular and singular perturbation problems Chandru *et al.* (2017),Kalaiselvan *et al.* (2019) and Lodhi *et al.* (2021).If the differential equation in which the highest order of the derivative term and the other term multiplied by another small positive parameter μ where, $0 < \mu \ll 1$ is known to be two parameters differential equation Gupta and Kaushik (2021),Mackey and Glass (1977),and Vaid et al. (2019). The aim of the study described in the given data is to propose an exponentially fitted finite difference scheme for solving singularly perturbed delay differential equations with two small parameters and a large delay. The scheme is designed to accurately capture the behavior of the solutions near the boundary layer and efficiently handle the delays involved. By developing a reliable and efficient numerical method, the study contributes to the advancement of numerical techniques for solving singularly perturbed delay differential equations with two small parameters and enhances our understanding and ability to model and analyze complex dynamic systems. A differential-difference equation also called a delay differential equation Chakravarthy et al.(2023),Phaneendra *et al.* (2014),Rai and Sharma (2011). is a type of functional differential equation.Where the system's evolution at a certain time depends not only on the present state of the system but also on the state of the system at an earlier time. The simplest form of A first-order delay differential equation with constant time delays can be expressed as:

$$u'(x) = f(x, u(x), u(x - \sigma_1), u(x - \sigma_2), ..., u(x - \sigma_k)).$$

There are different delay differential equation: such as small,large , advance and so no . Delay Singularly perturbed differential equations can be Space Shift and Time delay.

Science and technology have developed many practical problems based on Singularly perturbed differential equations,such as the mathematical boundary layer theory and the approximation of solutions to various problems described by differential equations, Physical phenomena in nature are modeled using differential equations.In singular perturbation problems,the regions of the solution that change rapidly in the interior of the domain are called Layers Derstine (1984),Debela and Duressa (2020). There are two different layers:boundary layers and interior layers.Most of the mathematical problems that arise in science and engineering are very hard and sometimes impossible to solve exactly. Thus, an approximation to a difficult mathematical problem is very important to make it easier to solve Singular perturbed delay differential equations.

1.2 Statement of the Problem

Different scholars have extensively researched the numerical solution for boundary value problems associated with singularly perturbed two-parameter delay differential equation convection diffusion reaction problems, such as:

- Singularly Perturbed Two-Parameter DEs (Phaneendra *et al.* (2017), Kusi *et al.* (2021), Lodhi *et al.* (2021), Chandru (2017))
- SP-DDEs (Vaid *et al.* (2019), Chakravarthy *et al.* (2017), Melesse *et al.* (2019), Swamy *et al.* (2015))
- SP-DDEs with Two Small Parameters (Chakravarthy *et al.* (2017), Debela and Duressa (2023), Hammachukiattikul *et al.* (2021), Prasad *et al.* (2022))

In a recent work, Kalaiselvan *et al.* (2019) proposed a fitted mesh method to effectively solve a challenging class of delay differential equations with two small parameters. However, a significant gap remains in the development of a uniformly convergent numerical method that can effectively solve these challenging problems. The numerical solutions is less than satisfactory for the problem under consideration. To explore the use of the exponential fitted operator method, coupled with a central difference formula, to develop a more accurate and fitted operator method for solving the problem effectively.

Therefore,this study is to bridge the gap in the development of accuracy and a uniformly convergent numerical method for solving the singularly perturbed two-parameter delay differential equation problem at hand. Specifically, this thesis aims to develop a more accurate and ε-uniformly convergent method for solving singularly perturbed convection-diffusion problems with a large delay of order one. This improved numerical treatment of the singularly perturbed boundary value problem under consideration will contribute to the advancement of the field and provide a more effective solution to this challenging class of problems.

Then the present study attempted to answer the following questions:

- How to describe the exponential fitted operator method for solving singularly perturbed two-parameter delay differential equation problems?
- To what extent does the proposed method converge?
- To what extent does the present method approximate the exact solution.

1.3 Objectives of the study

1.3.1 General Objective

The general objective of this study is to develop a robust numerical method for solving singularly perturbed two-parameter delay differential equations.

1.3.2 Specific Objectives

The study has the following specific objectives:

1. To formulate a numerical method for solving a specific type of singularly perturbed two-parameter delay differential equations.
2. To establish the convergence analysis of the proposed numerical scheme.
3. To investigate the accuracy of the method.

1.4 Significance of the Study

The significance of the study can be listed as follows:

- Developing a numerical method for solving singularly perturbed two-parameter delay differential equations is a significant contribution to the field of applied numerical mathematics.
- By studying the behavior of singularly perturbed delay differential equations, researchers can gain insights into the dynamics of complex systems with small perturbation parameters.
- The insights and techniques developed in this study can be generalized to other singularly perturbed systems beyond the specific problem addressed.
- To help the graduate student acquire research skills and scientific procedures.

1.5 Delimitation of the Study

Consider the singularly perturbed two-parameter large delay differential equation.

$$Lu \equiv \begin{cases} \varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) + c(x)u(x-1) = f(x), \quad x \in \Omega, \\ u(x) = \phi(x) \text{ on } [-1, 0], \\ u(2) = A, \end{cases}$$

where $\Omega = (0, 2)$ and $0 < \varepsilon \ll 1$, $0 < \mu < 1$ are two small parameters. $\bar{\Omega}_1 = (0, 1]$, $\Omega_1 = (0, 1)$ $\Omega_2 = (1, 2)$, $\Omega_0 = [-1, 0]$ and A is a given constant.

The coefficient functions $a(x)$, $b(x)$, $c(x)$, and $f(x)$ are sufficiently smooth on $\overline{\Omega} = [0, 2]$, and $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0$, $c(x) > 0$, and $(b - c)(x) \geq k > 0$. $\phi(x)$ is sufficiently smooth on $[-1, 0]$. The solution of this problem has boundary layers at the right of $x = 0$ and at the left of $x = 2$, and interior layers at $x = 1$. By using fitted operator method to increase the accuracy and uniform convergence of numerical methods for solving singularly perturbed delay differential equations.

Chapter 2

LITERATURE REVIEW

2.1 Singular perturbation Theory

The term "singular perturbation" was coined in the 1940s by Kurt Otto Friedrichs and Wolfgang R. Wasow.Perturbation theory is a field of study that examines the effects of small parameters in mathematical models, particularly in ordinary differential equations.In mathematics, a singular perturbation problem (SPP) refers to a problem that contains a small parameter that cannot be approximated by setting the parameter value to zero Kadalbajoo *et al.*(2003).Many theoretical and applied problems in various fields of science and technology, such as fluid mechanics, heat transfer, chemical kinetics, and electronics, give rise to boundary value problems (BVPs) for singularly perturbed differential equations. These problems exhibit a multi-scale nature, characterized by the presence of small parameters that lead to the formation of boundary layers in the solution. Analytical techniques alone are often insufficient to completely solve these problems.

Consequently, numerical methods play a critical role in obtaining useful insights into the solutions of singularly perturbed differential equations. These problems have solutions characterized by boundary layers or interior layers, and numerical simulations are indispensable for effectively studying their behavior. The development of science and technology has led to the emergence of practical problems that require sophisticated mathematical techniques. The approximation of solutions to differential equations involving small parameters, such as the boundary layer theory, have become increasingly complex and necessitates the use of asymptotic method O'Malley (1991).

Singularly perturbed problems arise in various applications, including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, civil engineering, and optimal control. These problems involve equations where the order is reduced by one if a small positive parameter multiplies the highest derivative term. This classification gives rise to boundary and interior layers in the solution, where the behavior changes abruptly in thin transition layers while varying slowly and regularly else where Chandru *et al.*(2017).Classical numerical methods struggle to provide accurate approximations for these problems.

Therefore, it is important to develop numerical methods that are a robust numerical method, meaning their accuracy does not depend on the perturbation parameter(s). The fitted operator method (FOM) and the fitted mesh method (FMM) are two widely used approaches for developing uniformly convergent numerical methods. FOM resolves the layers automatically due to the uniform mesh, while FMM uses finite difference schemes on specifically designed piecewise uniform meshes Miller *et al.* (1996) and Patidar (2008). Researchers have extensively studied the analytical and numerical solutions to these problems. The development of numerical methods for singularly perturbed problems initially focused on ordinary differential equations, as described in the first monograph on the subject by Doolan *et al.*(1980).

2.2 Singularly Perturbed Differential Equations.

Consider singularly perturbed two-point boundary value problems of the form

$$Lu \equiv \begin{cases} \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), & x \in \Omega, \\ u(0) = A, u(1) = B, \end{cases}$$

where $\Omega = (0,1)$ and $0 < \varepsilon \ll 1$, small parameters. A and B are given constants.

The coefficient functions $a(x)$,$b(x)$,and $f(x)$ are sufficiently smooth, with $a(x) \geq \alpha > 0$ and $b(x) \geq \beta > 0$, where $\alpha, \beta \in \mathcal{R}$.

Shishkinet *et al.* (2014) consider a Dirichlet problem for a singularly perturbed ordinary differential convection-diffusion equation, where a perturbation parameter (0,1) multiplies the highest-order derivative in the equation. The problem is approximated using the standard monotone finite difference scheme on a uniform grid. However, this scheme does not exhibit uniform convergence. Furthermore, even if it converges, it is not uniformly well-conditioned and stable when subjected to perturbations in the data of the discrete problem and/or computer perturbations.

Beckett and Mackenzie (2001). The convergence properties of a finite difference approximation for a singularly perturbed reaction-diffusion boundary value problem are investigated using a nonuniform grid. The construction of the grid is based on the equidistribution of a positive monitor function, which is a linear combination of a constant factor and a power of the second derivative of the solution. The analysis demonstrates how the monitor function can be carefully selected to ensure that the accuracy of the numerical approximation remains unaffected by the magnitude of the singular perturbation parameter. The application of equidistribution principles is commonly observed in various practical grid adaptation schemes, and our analysis provides valuable insights into the convergence behavior exhibited on such grids.

Debela (2021) presents an investigation into singularly perturbed differential equations of convection-diffusion type with nonlocal boundary conditions. The proposed numerical scheme combines the classical finite difference method for the boundary conditions and the exponential fitted operator method for the differential equations at the interior points. To assess the accuracy of the scheme, maximum absolute errors and rates of convergence are tabulated for various values of the perturbation parameter and mesh sizes in the numerical examples considered. The results demonstrate that the method maintains first-order accuracy regardless of the perturbation parameter ε.

Gupta and Kaushik (2021). A higher-order numerical approximation scheme is proposed for solving singularly perturbed reaction-diffusion boundary value problems. The scheme combines a fourth-order numerical difference method with the classical central difference method, applied on a non-equidistant grid. The non-equidistant grid is generated by equidistributing a non-negative monitor function. Theoretical and empirical error analyses are conducted, which demonstrate that the proposed scheme exhibits fourth-order uniform convergence with respect to the perturbation parameter.

Mohapatra *et al.* (2015) present an exponentially fitted finite difference method for solving singularly perturbed two-point boundary value problems that exhibit a boundary layer at the left or right end of the domain. To discretize the model equation, a fitting factor is introduced, and a finite difference scheme is applied to a uniform mesh. The tri-diagonal system resulting from the discretization is solved using the Thomas algorithm. The stability of the algorithm is thoroughly investigated.The analysis demonstrates that the proposed technique offers first-order accuracy, irrespective of the perturbation parameter.

2.3 Singularly Perturbed Two-Parameter Differential Equations.

Consider singularly perturbed Two-Parameter boundary value problems of the form

$$Lu \equiv \begin{cases} \varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega, \\ u(0) = A, u(1) = B, \end{cases}$$

where $\Omega = (0, 1)$ and $0 < \varepsilon \ll 1$, $0 < \mu < 1$ are two small parameters. A and B are given constants.

The coefficient functions $a(x)$, $b(x)$, and $f(x)$ are sufficiently smooth, with $a(x) \geq \alpha > 0$ and $b(x) \geq \beta > 0$, where $\alpha, \beta \in \mathcal{R}$.

Phaneendra *et al.* (2017) proposed a higher-order compact numerical method for solving a two-parameter singularly perturbed boundary value problem. The second-order singular perturbation problem is reformulated as a system of first-order ordinary differential equations. To solve this system, a seventh-order compact difference scheme is employed. The researchers solved various numerical examples and tabulated the maximum errors. A comparison with other methods from the existing literature was conducted to demonstrate the effectiveness of the proposed numerical scheme.

Kusi *et al.* (2021) introduced the quartic non-polynomial spline method for solving a singularly perturbed differential-difference equation with two parameters. The equation under consideration is transformed into an asymptotically equivalent differential equation, and the derivatives are approximated using finite difference approximation with the quartic non-polynomial spline method. The researchers also conducted a convergence analysis of the proposed method to establish its convergence properties.

Lodhi *et al.* (2021) developed a fourth-order compact finite difference method for numerically solving two-parameter singularly perturbed convection-diffusion boundary value problems. The authors utilized the fourth-order compact finite difference method on a uniform mesh, resulting in a tridiagonal linear system of equations. Through a matrix analysis approach, the convergence analysis of the proposed method was established, demonstrating that the method yields fourth-order convergence results.

Gracia (2006) presented a second-order monotone numerical method constructed for a singularly perturbed ordinary differential equation with two small parameters affecting the convection and diffusion terms. The monotone operator is combined with a piecewise uniform Shishkin mesh. An asymptotic error bound in the maximum norm is established theoretically, whose error constants are shown to be independent of both singular perturbation parameters.

Chandru (2017) presented A singularly perturbed second order ordinary differential equation having two parameters with a discontinuous source term is presented for numerical analysis. Theoretical bounds on the derivatives, regular and singular components of the solution are derived. A hybrid monotone difference scheme with the method of averaging at the discontinuous point is constructed on Shishkin mesh. Parameter-uniform error bounds for the numerical approximation are established.

Sahu *et al.* (2019) addressed a singularly perturbed delay differential equation that involves two small parameters. The paper proposed an upwind scheme on the fitted Shishkin mesh as the initial approach, which was shown to exhibit first-order convergence. Subsequently, a hybrid scheme was introduced, combining the midpoint upwind scheme in the outer region with the central difference scheme in the inner region. The error analysis was conducted for both schemes, revealing that the accuracy of the hybrid scheme was enhanced from first order to second order. This improvement in accuracy demonstrates the effectiveness of the proposed hybrid scheme for solving the singularly perturbed delay differential equation.

2.4 Singularly Perturbed Delay Differential Equations

A singularly perturbed delay differential equation (SPDDE) is a differential equation in which the highest derivative is multiplied by a small parameter and involves at least one shift term. Such problems arise frequently in the mathematical modeling of various physical and biological phenomena like optically bistable devices Derstine (1984) description of the human pupil reflex is a variety of models for physiological processes or disease variational problems in control theory and the first-exit time problem in the modeling of the activation of neuronal variability. This motivates the investigation of SPDDEs, and as a result, a series of papers have been devoted to this type of boundary value problem with a delay Subburayan *et al.* (2013).Amiraliyev *et al.* (1999) and Cimen (2020) study the boundary value problem (BVP) for the delay differential equation (DDE): Consider singularly perturbed Delay differential equation the form the interval and boundary conditions

$$Lu \equiv \begin{cases} \varepsilon u''(x) + a(x)u'(x) + b(x)u(x-\delta) = f(x), \quad x \in \Omega, \\ u(x) = \phi(x), \quad x \in \Omega_0, \\ u(1) = B, \end{cases}$$

where $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 = (0, \delta]$, $\Omega_2 = (\delta, 1]$, $\bar{\Omega} = [0, 1]$, $\Omega_0 = [-\delta, 0]$, $0 < \varepsilon \ll 1$ is the perturbation parameter, $a(x) \geq \alpha > 0$, $b(x)$, $f(x)$, and $\phi(x)$ are given sufficiently smooth functions satisfying certain regularity conditions to be specified, δ is a constant delay ($1 < 2\delta$), which is independent of ε, and B is a given constant. For small values of ε, the function $u(x)$ has a boundary layer near $x = 0$.

Vaid *et al.* (2019) presented a numerical technique for approximating the solution of a singularly perturbed delay differential equation. The prevalence of singularly perturbed delay differential equations in mathematical models of real-life applications has motivated researchers to develop numerical methods for their treatment. The proposed numerical technique is based on trigonometric cubic B-spline functions, in which derivatives are approximated as a linear sum of basis functions. The resulting matrix system is solved using the Thomas Algorithm. The convergence of the proposed method is thoroughly examined, and computational experiments are conducted on four examples to assess the effectiveness of the proposed scheme.

Chakravarthy *et al.* (2017) presented a singularly perturbed boundary value problem for a linear second-order delay differential equation. They observed that classical numerical methods are inadequate when applied to solve singularly perturbed problems in delay differential equations.They proposed an exponentially fitted finite difference scheme that overcomes the limitations of the corresponding classical methods. The stability of the scheme was thoroughly investigated, and its convergence properties were analyzed.

Melesse *et al.* (2019) presented an initial value method for solving a class of linear second-order singularly perturbed differential-difference equations that contain mixed shifts. The authors proposed a modification of the given problem by approximating the term involving the delay and advance parameters using a Taylor series expansion. This modification transformed the problem into an equivalent singularly perturbed problem. From the modified problem, two explicit initial value problems were derived, which are independent of the perturbation parameter. These problems are referred to as the reduced problem and the boundary layer correction problem. The authors solved these problems either analytically or numerically, and the obtained solutions were combined to provide an approximate solution to the original problem. Furthermore, the authors derived an error estimate for this method using the maximum norm.

Swamy *et al.* (2015) presented a computational method for solving singularly perturbed delay differential equations exhibiting twin layers or oscillatory behavior. The method involves transforming the original second-order singularly perturbed delay differential equation into an asymptotically equivalent first-order neutral-type delay differential equation. To solve the transformed equation, numerical integration and linear interpolation techniques are employed to obtain a tridiagonal system. The authors efficiently solve this tridiagonal system using the discrete invariant embedding algorithm.

2.5 Singularly Perturbed Delay Differential Equation Two Small Shifting Parameters.

Consider singularly perturbed Delay Differential Equation Two Small Parameters of the form

$$Lu \equiv \begin{cases} \varepsilon u''(x) + a(x)u'(x-\delta) + b(x)u(x+\gamma) + c(x)u(x) = f(x), \quad x \in \Omega, \\ u(x) = \phi(x), \quad x \in [-\delta, 0], \quad u(x) = \Phi(x), \quad x \in [1, 1+\gamma], \\ u(0) = \phi(0), \\ u(1) = \Phi(0), \end{cases}$$

where $\Omega = (0,1)$ and $0 < \varepsilon \ll 1$, is small parameter, δ and γ are given constants.Where,$\Omega = (0,1)$, $0 < \epsilon \ll 1$, and δ and γ are small shifting parameters, $0 < \delta \ll 1$, $0 < \gamma \ll 1$. The functions $a(x)$, $b(x)$, $c(x)$, $\phi(x)$, $\Phi(x)$, and $f(x)$ are sufficiently smooth in the interval $(0,1)$. The initial conditions are $u(0) = \phi(0)$ and $u(1) = \Phi(0)$.

Debela and Duressa (2022) investigated singularly perturbed differential-difference equations with mixed small shifts. The problem they addressed involved the presence of delay and advance parameters affecting the convection and reaction terms, respectively. The solution to this problem exhibited boundary layer behavior on either the left or right side of the domain, depending on the sign of the convection term. To approximate the terms involving delay and advance, the authors employed Taylor series expansion. Subsequently, they solved the resulting singularly perturbed boundary value problem using the fitted non-polynomial spline method.Stability and uniform convergence are established.

Rao *et al.* (2014) addressed the singularly perturbed boundary value problem for a linear second-order differential-difference equation of convection-diffusion type. In their paper, they proposed a numerical treatment for such problems by first employing Taylor's approximation to handle the term involving the small shift. To incorporate the effect of singular perturbations, a fitting parameter was introduced in a tridiagonal finite difference method, which was derived from the theory of singular perturbations. The resulting tridiagonal system was solved using the Thomas algorithm. The proposed method was carefully analyzed for convergence.

Hammachukiattikul *et al.* (2021) focused on a class of singularly perturbed advanced-delay differential equations of convection-diffusion type. To address this problem, the authors employed finite and hybrid difference schemes, which were implemented on a piecewise Shishkin mesh. They successfully established almost first- and second-order convergence for both the finite difference and hybrid difference methods. Additionally, an error estimate was derived using the discrete norm.

Prasad *et al.* (2022) introduced a novel computational method utilizing an exponential spline for solving a specific class of delay differential equations with a negative shift in the differentiated term. The proposed method demonstrates effectiveness when the shift parameter is of the order $O(\varepsilon)$. It effectively controls oscillations in the layer region of the solution. The numerical scheme incorporates a special type of mesh, and the parameter within the scheme is evaluated using the theory of singular perturbation. To show the efficiency of the proposed computational method and support the convergence analysis of the presented scheme,the maximum absolute errors and convergences of the numerical solutions are tabulated.

Kadalbajoo *et al.* (2004) presented a numerical method for solving boundary-value problems (BVPs) involving singularly perturbed differential-difference equations with a negative shift representing delay. This method is particularly useful for addressing BVPs associated with the expected first-exit time problem of the membrane potential in neuron models and variational problems in control theory. The paper thoroughly discusses the stability and convergence analysis of the proposed numerical method, providing valuable insights into its reliability and accuracy when applied to such BVPs.

2.6 Recent Development

Manikandan *et al.* (2014) introduce a parameter-uniform numerical method for solving a specific type of boundary value problem associated with singularly perturbed delay differential equations. The solution to this problem exhibits boundary layers at x = 0 and x = 2, as well as interior layers at x = 1. To solve the problem, they proposed a numerical method that combines a classical finite difference scheme with a piecewise uniform Shishkin mesh. The paper includes uniform asymptotic expansions, parameter-uniform numerical discretization, the shooting method, and thorough numerical analysis and convergence studies. The method has been proven to be first-order convergent in the maximum norm, uniformly in the perturbation parameter.The main contribution lies in the development of an accurate and efficient numerical method that remains stable and effective across a wide range of parameter values without the need for grid adaptation or parameter-specific adjustments.

In a recent development,

$$Lu \equiv \begin{cases} \varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) + c(x)u(x-1) = f(x), \quad x \in \Omega, \\ u(x) = \phi(x) \text{ on } [-1, 0], \\ u(2) = l. \end{cases}$$

For $0 < \mu < 1$, the solution of the above problem exhibits boundary layers which depend on both ε and μ, in particular on the ratio of μ^2 to ε as examined by O'malley (1967).

Kalaiselvan *et al.* (2019) propose a fitted mesh method to effectively solve a challenging class of delay differential equations with two small parameters. SP-DDEs are difficult to solve due to the presence of these small parameters, which cause rapid changes in the solution. The authors derive uniform asymptotic expansions that describe the solution's behavior across the entire domain, including boundary layers and the interior region. A numerical method composed of an upwind finite difference scheme applied to a piecewise uniform Shishkin mesh is suggested to solve the problem. This method is presented to be first-order convergent in the maximum norm uniformly in the perturbation parameters. They also provide a rigorous analysis of the proposed method, including error estimation, convergence properties, stability conditions, and accuracy. They conduct numerical experiments to validate the method, comparing the results with exact values.

Yet the obtained accuracy of the numerical solution is less accurate for the problem under consideration. Thus, it is necessary to produce more accuracy by using the exponential fitted operator method, applying a central difference formula, and finding a fitted operator method to solve the problem theoretically and experimentally for order of convergence. In this thesis,we use the same governing problem to solve it using the fitted operator method. The method was presented by employing an exponentially fitted operator to increase the accuracy and uniform convergence of numerical methods for solving singularly perturbed delay differential equations.

Chapter 3

MATERIALS AND METHODS

3.1 Study Area and Period

This study was conducted at Jimma University under the Department of Mathematics from January 2024 to June 2024.

3.2 Study Design

This study was employed mixed design :

- Documentary review design.
- Experimental design on singularly perturbed two-parameter delay differential equation problems.

3.3 Source of Information

The relevant sources of information for this study were:

- Relevant references book.
- Published articles.
- Articles Published in Reputable Journals (Research Gate).

3.4 Mathematical Procedure

The study had followed the following procedures in order to achieve the stated objectives:

1. Defining (or describing) the problem.
2. Describing the Properties of a Continuous Solution.
3. Discretizing the domain using uniform mesh for the defined problem.
4. Replacing the differential equation by the finite difference approximations,introducing the fitted operator,and central difference the system of algebraic equations.
5. Establishing the stability and convergence analysis of the formulated schemes.
6. Writing MATLAB code for the numerical method.
7. Validating the schemes by using numerical example.
8. Presenting the results using tables and graphs.

Chapter 4

DESCRIPTION OF THE METHODS, RESULTS AND DISCUSSION

4.1 Description of the Method

Consider singularly perturbed two-parameter large delay differential equation of the form.

$$\varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) + c(x)u(x-1) = f(x) \qquad x \in \Omega, \tag{4.1}$$

subject to with the interval and boundary conditions

$$u(x) = \phi(x) \quad \text{on } [-1, 0], \quad u(2) = A.$$

where $\Omega = (0, 2)$ and $0 < \varepsilon \ll 1$, $0 < \mu < 1$ are two small parameters. $\bar{\Omega}_1 = (0, 1]$, $\Omega_1 = (0, 1)$,$\Omega_2 = (1, 2)$, $\bar{\Omega}_2 = [1, 2)$,$\bar{\Omega} = [0, 2]$,$\Omega_* = \Omega_1 \cup \Omega_2$,$\Omega_0 = [-1, 0]$ and A is a given constant.

Eq.(4.1) is written in the form of

$$\varepsilon u''(x) + p(x)u'(x) - b(x)u(x) + c(x)u(x-1) = f(x) \qquad x \in \Omega, \tag{4.2}$$

where $p(x) = \mu a(x)$.

From Eq.(4.2) the values of $u(x-1)$ are known for the domain $\bar{\Omega}_1 = (0,1]$and unknown for the $\Omega_2 = (1,2)$ due to the large delay at x = 1. Thus we have to treat the problem at $\bar{\Omega}_1$ and Ω_2 separately.

We can write Eq.(4.2)in operator form as:

$$Lu(x) = R(x) \tag{4.3}$$

where

$$Lu(x) = \begin{cases} L_1u(x) = \varepsilon u''(x) + p(x)u'(x) - b(x)u(x), & x \in \bar{\Omega}_1, \\ L_2u(x) = \varepsilon u''(x) + p(x)u'(x) - b(x)u(x) + c(x)u(x-1), & x \in \Omega_2, \end{cases} \tag{4.4}$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), & x \in \bar{\Omega}_1, \\ f(x), & x \in \Omega_2. \end{cases} \tag{4.5}$$

with interval and boundary conditions:

$$\begin{cases} u(x) = \phi(x), & x \in [-1,0], \\ u(1^-) = u(1^+), u'(1^-) = u'(1^+), \\ u(2) = A, \end{cases} \tag{4.6}$$

where $u(1^-)$ and $u(1^+)$ denoted the left and right limits of $u(x)$ at $x = 1$, respectively.

4.2 Properties of Continuous Solution

Lemma 1(Continuous Solution)[20]

Eq.(4.2)and Eq.(4.3) are equivalent if $u(1^-) = u(1) = u(1^+)$.

Proof : It is obvious that a solution of Eq.(4.2) is also a solution of Eq.(4.4).

Let $u(x)$ be a solution of Eq.(4.5).

Then $u(1^-) = u(1^+)$, $u'(1^-) = u'(1^+)$ and

$$\varepsilon u''(1^-) + p(1^-)u'(1^-) - b(1^-)u(1^-) = R(1^-) \tag{4.7}$$

$$\varepsilon u''(1^+) + p(1^+)u(1^+) - b(1^+)u(1+) + c(1^+)u(0^+) = f(1^+) \tag{4.8}$$

Eq.(4.7) and Eq.(4.8) lead to $u''(1^-) = u''(1^+)$ and hence $u(x) \in (\bar{\Omega})$.

Using the continuity of $u(x)$, $u'(x)$, $u''(x)$ and $f(x)$ at $x = 1$,

$\varepsilon u''(1) + p(1)u'(1) - b(1)u(1) + c(1)u(0) = f(1)$

Therefore, $\varepsilon u''(x) + p(x)u'(x) - b(x)u(x) + c(x)u(x-1) = f(x)$, $x \in \Omega$.

Hence, $u(x)$ is a solution of Eq.(4.2).

Lemma 2 (Minimum Principle)[20]: Let $\psi(x)$ be any function in $C^2(\bar{\Omega})$ such that $\psi(0) \geq 0$, $\psi(2) \geq 0$, $L_1\psi(x) \leq 0$ $\forall x \in \Omega_1$ and $L_2\psi(x) \leq 0$ $\forall x \in \bar{\Omega}_2$, then $\psi(x) \geq 0$ for all $x \in \overline{\Omega}$.

Proof : Let x^* be such that $\psi(x^*) = \min_{x\in[0,2]} \psi(x)$.

Suppose $\psi(x^*) < 0$; then $x^* \notin \{0, 2\}$, $\psi'(x^*) = 0$ and $\psi''(x^*) \geq 0$.

Claim 1: $x^* \notin (0, 1)$:

If $x^* \in (0, 1)$, then $L_1\psi(x^*) = \varepsilon\psi''(x^*) + p(x^*)\psi'(x^*) - b(x^*)\psi(x^*) > 0$,

Contradiction to the assumption that $L_1\psi \leq 0$ on $(0, 1)$.

Hence $x^* \notin (0, 1)$.

Claim 2: $x^* \notin [1, 2)$:

If $x^* \in [1, 2)$, then $L_2\psi(x^*) \geq \varepsilon\psi''(x^*) + p(x^*)\psi'(x^*) - (b(x^*) - c(x^*))\psi(x^*) > 0$,

contradiction to the assumption that $L_2\psi \leq 0$ on $[1, 2)$.

Hence $x^* \notin [1, 2)$.

Thus, $\psi(x) \geq 0$, on $[0, 2]$.

Hence, the proof of the lemma.

Lemma 3 (Stability Result)[20]: Let $\psi(x) \in C^2(\bar{\Omega})$, then for $x \in \bar{\Omega}$, we have

$$|\psi(x)| \leq \max\{|\psi(0)|, |\psi(2)|, \frac{1}{k}\|L_1\psi\|_{\bar{\Omega}_1}, \frac{1}{k}\|L_2\psi\|_{\bar{\Omega}_2}\}.$$

Proof:

Let $M = \max\{|\psi(0)|, |\psi(2)|, \frac{1}{k}\|L_1\psi\|_{\bar{\Omega}_1}, \frac{1}{k}\|L_2\psi\|_{\bar{\Omega}_2}\}$.

Consider the functions $\theta^{\pm}(x) = M \pm \psi(x)$.

Clearly, $\theta^{\pm} \in C^2(\bar{\Omega})$, $\theta^{\pm}(0) \geq 0$, and $\theta^{\pm}(2) \geq 0$.

Case 1:For $x \in (0,1)$, $L_1\theta^{\pm}(x) = -b(x)M \pm L_1\psi(x) \leq 0$. and

Case 2:For $x \in [1,2)$, $L_2\theta^{\pm}(x) = -b(x)M + c(x)M \pm L_2\psi(x) \leq 0$.

Using Lemma 2, we have $|\psi(x)| \leq M$ for $x \in \bar{\Omega}$.

Corollary If $u(x)$ is the solution of Eq.(4.1), then

$$|u(x)| \leq \max\left\{|u(0)|, |u(2)|, \frac{1}{k}\|f\|\right\}. \tag{4.9}$$

Theorem[20]: Assuming that $a(x)$, $b(x)$, $c(x)$, and $f(x)$ are in $C^2(\bar{\Omega})$, the derivatives of the solution $u(x)$ of Eq.(4.1) satisfy the following bounds:

$$\|u^{(k)}\| \leq \frac{C}{(\sqrt{\varepsilon})^k}\left(1 + \left(\frac{C_\mu}{\sqrt{\varepsilon}}\right)^k\right)\max\{\|u\|, \|f\|\}, \quad \text{for } k = 1, 2,$$
$$\|u^{(3)}\| \leq \frac{C}{(\sqrt{\varepsilon})^3}\left(1 + \left(\frac{C_\mu}{\sqrt{\varepsilon}}\right)^3\right)\max\{\|u\|, \frac{1}{\varepsilon}\|f'\|\},$$

where the constant C is independent of ε and C_μ.

Proof:

Let $N_x = (p, p + \sqrt{\varepsilon})$ for $x \in [0,2]$, where $0 \leq p \leq x \leq p + \sqrt{\varepsilon} \leq 2$. By the mean value theorem, there exists a $y \in N_x$ such that

$$u'(y) = \frac{u(p + \sqrt{\varepsilon}) - u(p)}{\sqrt{\varepsilon}} \leq \frac{2\|u\|}{\sqrt{\varepsilon}}.$$

Also,

$$u'(x) = u'(y) + \int_y^x u''(s)\,ds.$$

Substituting for $u''(s)$ from Eq.(4.1) and using the bounds of $a(x)$, $b(x)$, $c(x)$, and $f(x)$ on $(0,2)$, we obtain

$$\|u'\| \leq C\left(\frac{1}{\sqrt{\varepsilon}} + \frac{C_\mu}{\varepsilon}\right)\max\{\|u\|, \|f\|\}.$$

Again, from Eq.(4.1),

$$\|u''\| \leq C\left(\frac{1}{\varepsilon} + \frac{C_\mu^2}{\varepsilon^2}\right)\max\{\|u\|, \|f\|\}.$$

Differentiating Eq.(4.1) once and using the bounds of u, u', and u'', we get

$$\|u^{(3)}\| \leq \frac{C}{(\sqrt{\varepsilon})^3}\left(1 + \left(\frac{C_\mu}{\sqrt{\varepsilon}}\right)^3\right)\max\{\|u\|, \|f\|\} + \frac{1}{\varepsilon}\|f'\|$$

Lemma 4 (Estimates of smooth)[20]

Let $\psi \in C^1\bar{\Omega}_1$ and $\psi(1) \geq 0$, then $L_3\psi \leq 0$ on $[0,1)$ implies that $\psi \geq 0$ on $[0,1]$.

proof

Let $x^* \in [0,1]$ be such that $\psi(x^*) = \min_{x\in\bar{\Omega}_1}\psi(x)$.

Suppose $\psi(x^*) < 0$; then $x^* \neq 1$ and $\psi'(x^*) \geq 0$.

If $x^* \in [0,1)$, then $L_3\psi(x^*) = p(x^*)\psi'(x^*) - b(x^*)\psi(x^*) > 0$,

which contradicts the assumption that $L_3\psi \leq 0$ on $[0,1)$.

Thus, $\psi(x) \geq 0$ on $[0,1]$.

4.3 Numerical Scheme Formulation

The linear ordinary differential equation in Eq.(4.1) cannot, in general, be solved analytically because of the dependence of $a(x)$, $b(x)$, and $c(x)$ on the spatial coordinate x. We divide the interval Ω= $(0,2)$ into $2N$ equal parts (uniform) with a constant mesh length $h = \frac{2-0}{2N} = \frac{1}{N} = N^{-1}$. If we consider the interval $x \in \bar{\Omega}_1$, and $x \in \Omega_2$ .The domain $(0,1]$ is discretized into N equal subintervals, each of length h. Let $0 = x_0 < x_1 < x_2 < \ldots < x_N = 1$ be the points such that We apply an exponentially fitted operator finite difference method (FOFDM). From Eq. (4.4) and Eq. (4.5), we have

$$L_1u(x) = \begin{cases} \varepsilon u''(x) + p(x)u'(x) + b(x)u(x) = R(x), & x \in (0,1), \\ u(0) = \phi(0), \end{cases} \tag{4.10}$$

where $R(x) = f(x) - c(x)\phi(x-1)$.

To find the numerical solution of Eq. (4.10), we use the theory used in the asymptotic method for solving singularly perturbed BVPs. In the considered case, the boundary layer is on the left side of the domain, i.e., near $x = 0$. From the theory of singular perturbations given by O'Riordan (1986) and O'Malley(1991) we get an approximation (up to first order) of the asymptotic solution in the form

$$u(x) = u_0(x) + \frac{p(0)}{p(x)}\left(u(0) - u_0(0)\right)\exp\left(-\int_0^x \frac{p(x)}{\varepsilon} - \frac{b(x)}{p(x)}\right)dx + O(\varepsilon),$$

By using the Taylor series around $x = 0$ for $p(x)$ and $b(x)$ and simplifying, we obtain

$$u(x) = u_0(x) + (\phi(0) - u_0(0))\exp\left(-\frac{p^2(0) - \varepsilon b(0)}{\varepsilon p(0)}\right)(x) + O(\varepsilon). \tag{4.11}$$

where $u_0(x)$ is the solution of the reduced problem (obtained by setting $\varepsilon = 0$) of Eq. (4.10), which is given by

$$p(x)u'(x) + b(x)u(x) = R(x). \tag{4.12}$$

By considering a small enough h, the discretized form of Eq. (4.11) becomes

$$u(ih) = u_0(ih) + (u(0) - u_0(0))\exp\left(-\frac{p^2(0) - \varepsilon b(0)}{p(0)}\left(\frac{1}{\varepsilon} - i\rho\right)\right). \tag{4.13}$$

where $\rho = \frac{h}{\varepsilon}, \quad h = \frac{1}{N}$.

$$u_{i\pm1} = u_0((i \pm 1)h) + (u(0) - u_0(0))\exp\left(-\frac{p^2(0) - \varepsilon b(0)}{p(0)}\left(\frac{1}{\varepsilon} - (i \pm 1)\rho\right)\right).$$

Using Taylors series approximation for $u_0((i+1)h)$ and $u_0((i-1)h)$up to

first order, we obtain.

$$\begin{cases} u_{i+1} = u_0((ih) + (u(0) - u_0(0)) \exp\left(-\frac{p^2(0)-\varepsilon b(0)}{p(0)}\left(\frac{1}{\varepsilon} - (i+1)\rho\right)\right), \\ u_i = u_0(ih) + (u(0) - u_0(0)) \exp\left(-\frac{p^2(0)-\varepsilon b(0)}{p(0)}\left(\frac{1}{\varepsilon} - i\rho\right)\right), \\ u_{i-1} = u_0((ih) + (u(0) - u_0(0)) \exp\left(-\frac{p^2(0)-\varepsilon b(0)}{p(0)}\left(\frac{1}{\varepsilon} - (i-1)\rho\right)\right). \end{cases} \tag{4.14}$$

From Eq.(4.14) we can getting

$$\begin{cases} u_{i+1} - 2u_i + u_{i-1} = \alpha \exp\left(\theta\left(\frac{1}{\varepsilon} - i\rho\right)\right)[2\cosh(\theta\rho) - 2], \\ u_{i+1} - u_{i-1} = \alpha \exp\left(\theta\left(\frac{1}{\varepsilon} - i\rho\right)\right)[-2\sinh((\theta\rho)]. \end{cases} \tag{4.15}$$

$$\text{where} \quad \alpha = u(0) - u_0(1), \quad \theta = -\frac{p^2(0) - \varepsilon b(0)}{p(0)}.$$

To handle the fitting factor (artificial viscosity) caused by the perturbation parameter, the term containing the perturbation parameter is multiplied by the exponentially fitting factor $\sigma(\rho)$ as

$\varepsilon u''(x) + p(x)u'(x) - b(x)u(x) = R(x)$

$$\varepsilon\sigma(\rho)u''(x_i) + p(x)u'(x_i) - b(x_i)u(x_i) = R(x_i) \tag{4.16}$$

Next, on a uniform mesh points $\overline{\Omega}^N = \{x_i\}_{i=0}^N$ and with $h = x_{i+1} - x_i$, using the difference approximations:

$$\begin{cases} D^0u(x_i) &= \frac{u(x_{i+1})-u(x_{i-1})}{2h} + \tau_3. \\ D^+D^-u(x_i) &= \frac{u(x_{i-1})-2u(x_i)+u(x_{i+1})}{h^2} + \tau_4. \end{cases} \tag{4.17}$$

where $\tau_3 = -\frac{h^2}{6}u^{(3)}(x_i)$ and $\tau_4 = -\frac{h^2}{12}u^{(4)}(x_i)$. When we apply a central difference formula on Eq. (4.16), it takes the form

$$\varepsilon\sigma(\rho)\left(D^+D^-u(x_i)\right) + p(x_i)\left(D^0u(x_i)\right) - b(x_i)u(x_i) = R(x_i). \tag{4.18}$$

Let U_i be an approximate solution of $u(x)$ and R_i be an approximation of $R(x)$ at grid point x_i, then we write the numerical scheme for Eq. (4.18) in difference operator form as

$$L^N U_i = R_i. \tag{4.19}$$

with boundary conditions $U_0 = \phi(0)$, where

$$L^N U_i = \varepsilon\sigma(\rho)\left(\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\right) + p(x_i)\left(\frac{U_{i+1} - U_{i-1}}{2h}\right) - b(x_i)U_i = R_i. \tag{4.20}$$

Now to find the fitted operator at the left end of boundary layer at $x = 0$ The multiplication of Eq. (4.20) by h, then tending to zero with h and truncating the term $(R_i + b(x_i)U_i)h$, results in

$$\frac{\sigma(\rho)}{\rho}\left(U_{i+1} - 2U_i + U_{i-1}\right) + \frac{p_i}{2}\left(U_{i+1} - U_{i-1}\right) = 0. \tag{4.21}$$

where $\quad \rho = \frac{h}{\varepsilon}, \quad h = \frac{1}{N}$.

By substituting the results of Eq. (4.15) an into Eq. (4.21) and simplification, the exponential fitting factor is obtained as

$$\sigma_1(\rho) = \frac{\rho p(0)}{2}\coth\left(\frac{\rho p(0)}{2}\right). \tag{4.22}$$

In a similar minor to find the fitted operator at the right end of the boundary layer at $x = 2$.

$$\sigma_2(\rho) = \frac{\rho p(2)}{2}\coth\left(\frac{\rho p(2)}{2}\right). \tag{4.23}$$

Assume that $\overline{\Omega}$ denotes the partition of $[0, 2]$ into $2N$ subintervals such that $0 = x_0, x_1, x_2, \ldots, x_N = 1$ and $x_{N+1}, x_{N+2}, \ldots, x_{2N} = 2$ with $x_i = ih$, $h = \frac{2}{2N} = \frac{1}{N}$, $i = 0, 1, 2, \ldots, 2N$. Then to developed the scheme of at $\bar{\Omega}_1$ and Ω_2 separately.

Case (1): Consider Eq.(4.4) and (4.5) on the domain $(0, 1]$, which is given by

$$\varepsilon\sigma(\rho)u''(x) + p(x)u'(x) - b(x)u(x) = f(x) - c(x)\phi(x - 1). \tag{4.24}$$

By discretized Eq.(24)

$$\varepsilon\sigma(\rho)u''(x_i) + p(x_i)u'(x_i) - b(x_i)u(x_i) = f(x_i) - c(x_i)\phi(x_i - 1). \tag{4.25}$$

$$\varepsilon\sigma(\rho)\left(\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\right) + p_i\left(\frac{U_{i+1} - U_{i-1}}{2h}\right) - b_iU_i = f_i - c_i\phi(x_i - x_N). \tag{4.26}$$

$$\left(\frac{\varepsilon\sigma(\rho)}{h^2} - \frac{p_i}{2h}\right)U_{i-1} - \left(\frac{2\varepsilon\sigma(\rho)}{h^2} + b_i\right)U_i + \left(\frac{\varepsilon\sigma(\rho)}{h^2} + \frac{p_i}{2h}\right)U_{i+1} = f_i - c_i\phi(x_i - x_N). \tag{4.27}$$

for $i = 1, 2, \ldots, N$.

The numerical scheme in Eq. (4.27) can be written as three-term recurrence relations

$$E_iU_{i-1} + F_iU_i + G_iU_{i+1} = H_i, \quad i = 1, 2, \ldots, N. \tag{4.28}$$

where

$$E_i = \frac{\varepsilon\sigma(\rho)}{h^2} - \frac{p_i}{2h}, \quad F_i = -\left[\frac{2\varepsilon\sigma(\rho)}{h^2} + b_i\right], \quad G_i = \frac{\varepsilon\sigma(\rho)}{h^2} + \frac{p_i}{2h} \quad \text{and} \quad H_i = f_i - c_i\phi(x_i - x_N).$$

Case (2): Consider Eq. (4.4) and (4.5) on the domain $\Omega_2 = (1, 2)$ using the exponentially fitted finite difference method, which is given by

$$\varepsilon\sigma(\rho)u''(x) + p(x)u'(x) - b(x)u(x) + c(x)u(x)(x - 1) = f(x). \tag{4.29}$$

By discretized Eq.(29)

$$\varepsilon\sigma(\rho)u''(x_i) + p(x_i)u'(x_i) - b(x_i)u(x_i) + c(x_i)u(x_i)(x_i - 1) = f(x). \tag{4.30}$$

$$\varepsilon\sigma(\rho)\left(\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\right) + p_i\left(\frac{U_{i+1} - U_{i-1}}{2h}\right) - b_iU_i + c_iU(x_i - 1) = f_i. \tag{4.31}$$

$$\left(\frac{\varepsilon\sigma(\rho)}{h^2} - \frac{p_i}{2h}\right)U_{i-1} - \left(\frac{2\varepsilon\sigma(\rho)}{h^2} + b_i\right)U_i + \left(\frac{\varepsilon\sigma(\rho)}{h^2} + \frac{p_i}{2h}\right)U_{i+1} + c_iU(x_i - 1) = f_i. \tag{4.32}$$

Similarly, this equation can be written as

$$E_iU_{i-1} + F_iU_i + G_iU_{i+1} + M_i = H_i, \quad i = N+1, N+2, \dots, 2N-1. \tag{4.33}$$

where

$$E_i = \frac{\varepsilon\sigma(\rho)}{h^2} - \frac{p_i}{2h}, \quad F_i = -\left[\frac{2\varepsilon\sigma(\rho)}{h^2} + b_i\right], \quad G_i = \frac{\varepsilon\sigma(\rho)}{h^2} + \frac{p_i}{2h}$$

$$M_i = c_iU(x_i - 1) \quad \text{and} \quad H_i = f_i.$$

Therefore, on the whole domain $\bar{\Omega} = [0, 2]$, the basic schemes to solve Eq. (4.1) are the schemes given in Eq. (4.28) and Eq. (4.33) together, which can be solved by using the Gaussian elimination method or inverse method. But the recommended method is the Gaussian elimination method, but the other methods. very costly and time-killer, and also with the local truncation error of τ_3.

The matrix representation of the scheme on the whole domain of $\Omega = (0, 2)$.

$$\begin{bmatrix}
F_1 & G_1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\
E_2 & F_2 & G_2 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & E_3 & F_3 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & G_{N-1} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & F_N & G_N & 0 & \cdots & 0 & 0 \\
M_1 & 0 & 0 & \cdots & E_{N+1} & F_{N+1} & G_{N+1} & \cdots & 0 & 0 \\
0 & M_2 & 0 & \cdots & 0 & F_{N+2} & G_{N+2} & \cdots & 0 & 0 \\
0 & 0 & M_3 & \cdots & 0 & 0 & F_{N+3} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \cdots & \ddots & \vdots \\
0 & \ddots & 0 & \cdots & M_{N-2} & 0 & \vdots & \ddots & F_{2N-2} & G_{2N-2} \\
0 & \cdots & 0 & \cdots & 0 & M_{N-1} & \cdots & 0 & E_{2N-1} & F_{2N-1}
\end{bmatrix}
\begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \\ u_N \\ u_{N+1} \\ u_{N+2} \\ u_{N+3} \\ \vdots \\ u_{2N-2} \\ u_{2N-1}
\end{bmatrix}
=
\begin{bmatrix}
H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_{N-2} \\ H_{N-1} \\ H_N \\ H_{N+1} \\ H_{N+2} \\ H_{N+3} \\ \vdots \\ H_{2N-2} \\ H_{2N-1}
\end{bmatrix}$$

4.4 Uniform Convergence Analysis

The discrete scheme corresponding to the original Eq. (4.4) and (4.5) is as follows:

For $i = 1, 2, 3, \ldots, N$.

$$L_1^N U_i = f_i - c_i \phi_{i-N}. \tag{4.34}$$

For $i = N+1, N+2, \ldots, 2N-1$.

$$L_2^N U_i = f_i, \tag{4.35}$$

subject to the boundary conditions:

$$U_i = \phi_i, \quad i = -N, -N+1, \ldots, 0. \tag{4.36}$$

$$U_{2N} = A, \tag{4.37}$$

where

$$\begin{cases} L_1^N U_i = \varepsilon D^+ D^- U_i + p(x_i) D^0 U_i - b(x_i) U_i \\ L_2^N U_i = \varepsilon D^+ D^- U_i + p(x_i) D^0 U_i - b(x_i) U_i + c(x_i) U_{i-N}. \end{cases} \tag{4.38}$$

Lemma 5: (Discrete Minimum Principle)[20]

Assume that the mesh function $\psi(x_i)$ satisfies $\psi(x_0) \geq 0$ and $\psi(x_{2N}) \geq 0$.

Then $L_1^N(\psi(x_i)) \leq 0\ \forall i \in \Omega_1$and $L_2^N \psi(x_i) \leq 0\ \forall i \in \bar{\Omega}_2$,

implies that Then $\psi(x_i) \geq 0$, $\forall i \in \bar{\Omega}$.

Proof : Let k be such that $\psi(x_k) = \min_{i \in \bar{\Omega}} \psi(x_i)$. and

Suppose $\psi(x_k) < 0$; then $k \notin \{0, 2\}$, $\psi(x_{k+1}) - \psi(x_k) \geq 0$ and $\psi(x_k) - \psi(x_{k-1}) \leq 0$.

Also,$L_1^N \psi(x_k) \geq -b(x_k)\psi(x_k) > 0$,if $k \in \Omega_1$ and

$L_2^N \psi(x_k) \geq \psi(x_k)\left(c(x_k) - b(x_k)\right) \geq 0$ if $k \in \bar{\Omega}_2$,

as $b(x) - c(x) > 0, x \in \bar{\Omega}$.

Therefore,$\psi(x_k) \geq 0$ and

hence, $\psi(x_i) \geq 0$, for $i \in \bar{\Omega}$.

Hence, the proof of the lemma is finished.

Remark: The above problem Eq.(4.34) has a solution[38],the solution is unique due to the above Minimum principle.

Lemma 6(Discrete stability result):[20]

Let $|\psi(x_i)|$ be any mesh function,then

$|\psi(x_i)| \leq C \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i\in\Omega_1} |L_1^N \psi(x_i)|, \max_{i\in\bar{\Omega}_2} |L_2^N \psi(x_i)|\}$

Proof Let us define the barrier function

Let us use the notation $C = \min\{\psi(x_i) : x_i \in \bar{\Omega}^{2N}\}$. Then,

$$\theta^{\pm}(x_i) = CM \pm \psi(x_i), \forall x_i \in \bar{\Omega}^{2N} \tag{4.39}$$

where

$$M = \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i\in\Omega_1^{2N}\cup\Omega_2^{2N}} |L^N \psi(x_i)|\}.$$

it is clear that $\theta^{\pm}(x_0) \geq 0$ and $\theta^{\pm}(x_{2N}) \geq 0$,

$$L_1^N \theta^{\pm}(x_i) \geq 0,\ \forall x_i \in \Omega_1^{2N}\ ,$$
$$L_2^N \theta^{\pm}(x_i) \geq 0,\ \forall x_i \in \Omega_2^{2N} \geq 0\ ,$$
$$(\theta^{\pm})'(x_{2N}) \leq 0.$$

Thus, using Lemma 3, $\theta^{\pm}(x_i) \geq 0$, $\forall x_i \in \bar{\Omega}^{2N}$

We have proved above that the discrete operator L^N satisfies the maximum principle. Next, we analyze the uniform convergence of the method.

Theorem 2:[38] Let $u(x_i)$ and U_i be the exact solution of Eq. (4.1) and numerical solutions of Eq.(4.18) respectively. Then, for a sufficiently large N, the following parameter uniform error estimate holds:

$$|L^N(u(x_i) - U_i)| \leq \frac{CN^{-2}}{N^{-1} + \varepsilon} + \left(1 + \varepsilon^{-3} \exp\left(-\frac{p(1-x_i)}{\varepsilon}\right)\right). \tag{4.40}$$

Proof Let us consider the local truncation error defined as
$L^N(u(x_i) - U_i) = \varepsilon\sigma(\rho)(u''(x_i) - D^+D^-u(x_i)) + p(x_i)(u'(x_i) - D^0u(x_i))$.

$$= \varepsilon \left[\frac{\rho p(0)}{2} \coth\left(\frac{\rho p(0)}{2}\right) - 1\right] D^+D^-u(x_i) + \varepsilon(u''(x_i) - D^+D^-u(x_i)) + p(x_i)(u'(x_i) - D^0u(x_i)). \tag{4.41}$$

where $\sigma(\rho) = p(0)\frac{\rho}{2} \coth\left(p(0)\frac{\rho}{2}\right)$ and $\rho = \frac{N^{-1}}{\varepsilon}$.

Since $|z\coth(z) - 1| \leq z^2$ holds if $z \neq 0$ and also $|z\coth(z) - 1| \leq z$ if $z > 0$ values. Now, for $z > 0$, C_1 and C_2 are constants, and we have $|z\coth(z) - 1| \leq C_1 z^2$, $z \leq 1$. Similarly, for $z \to \infty$, since $\lim_{z\to\infty} \coth(z) = 1$, $|z\coth(z) - 1| \leq C_1 z$ is given. In general, for all $z > 0$, we can write

$$C_1 \frac{z^2}{(z+1)} \leq z\coth(z) - 1 \leq C_2 \frac{z^2}{(z+1)}. \tag{4.42}$$

Implying that

$$\varepsilon \left[\frac{\rho p(0)}{2} \coth\left(\frac{\rho p(0)}{2} \right) - 1 \right] \leq \varepsilon \left(\frac{(N^{-1}/\varepsilon)^2}{N^{-1}/\varepsilon + 1}. \right) = \frac{N^{-2}}{N^{-1} + \varepsilon}. \tag{4.43}$$

Using Taylor series expansion, we can rewrite $u(x_{i-1})$ and $u(x_{i+1})$ in terms of the values and derivatives of $u(x_i)$ as

$$\begin{cases} u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2!}u''(x_i) - \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(x_i) + O(h^5). \\ u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2!}u''(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(x_i) + O(h^5). \end{cases}$$

We obtain the bounds for the second-order derivatives as

$$\begin{cases} |D^+D^-u(x_i)| \leq C|u''(x_i)|. \\ |u''(x_i) - D^+D^-u(x_i)| \leq CN^{-2}|u^{(4)}(x_i)|. \end{cases} \tag{4.44}$$

Similarly, for the first derivative term

$$|u'(x_i) - D^0u(x_i)| \leq CN^{-2}|u^{(3)}(x_i)|, \tag{4.45}$$

where $|u^{(k)}(x_i)| = \sup_{x_i \in (x_0, x_N)} |u^{(k)}(x_i)|$, $k = 2, 3, 4$. Using the bounds in Eq.(4.40) and Eq.(4.41), we obtain

$$|L^N(u(x_i) - u_i)| \leq C\frac{N^{-2}}{N^{-1} + \varepsilon}|u''(x_i)| + \varepsilon CN^{-2}\left|u^{(4)}(x_i)\right| + CN^{-2}\left|u^{(3)}(x_i)\right|.$$

$$\leq C\frac{N^{-2}}{N^{-1} + \varepsilon}|u''(x_i)| + CN^{-2}\left(\varepsilon\left|u^{(4)}(x_i)\right| + \left|u^{(3)}(x_i)\right|\right).$$

Now, using the bounds for the derivatives of the solution in lemma 4 and the assumption

$\varepsilon \leq N^{-1}$, Eq. (4.43), we have

$$\begin{aligned}
\left|L^N(u(x_i)-U_i)\right| \leq & \frac{CN^{-2}}{N^{-1}+\varepsilon}\left(1+\varepsilon^{-2}\exp\left(\frac{-px_j}{\varepsilon}\right)\right) \\
& + CN^{-2}\left[\varepsilon\left(1+\varepsilon^{-4}\left(\frac{-px_j}{\varepsilon}\right) + \left(1+\epsilon^{-3}\exp\left(\frac{-px_j}{\varepsilon}\right)\right]\right]. \\
& \leq \frac{CN^{-2}}{N^{-1}+\varepsilon}\left(1+\varepsilon^{-2}\exp\left(\frac{-px_j}{\varepsilon}\right)\right) \\
& + CN^{-2}\left[\varepsilon\left(1+\varepsilon^{-4}\left(\frac{-px_j}{\varepsilon}\right)\right) + \left(1+\epsilon^{-3}\exp\left(\frac{-px_j}{\varepsilon}\right)\right)\right].
\end{aligned}$$

which simplifies to

$$\left|L^N(u(x_i)-U_i)\right| \leq \frac{CN^{-2}}{N^{-1}+\varepsilon}\left(1+\epsilon^{-3}\exp\left(\frac{-px_j}{\varepsilon}\right)\right) \quad \text{since } \varepsilon^{-3} \geq \varepsilon^{-2} \tag{4.46}$$

Lemma 7: [7] For a fixed mesh and for $\varepsilon \to 0$, the following holds:

$$\lim_{\varepsilon\to 0}\max_{1\leq j\leq N-1}\frac{\exp\left(\frac{-px_j}{\varepsilon}\right)}{\varepsilon^m} = 0; \quad m = 1, 2, 3, \ldots$$

$$\lim_{\varepsilon\to 0}\max_{1\leq j\leq N-1}\frac{\exp\left(\frac{-p(1-x_j)}{\varepsilon}\right)}{\varepsilon^m} = 0; \quad m = 1, 2, 3, \ldots$$

where $x_i = ih$, $h = \frac{1}{N}$, $i = 1, 2, 3, \ldots, N-1$.

Proof:[7]

Theorem 3: Let $u(x_i)$ and U_i be the exact solution of Eq. (4.1) and numerical solutions of Eq. (4.38) respectively. Then, the following error bound holds:

$$\sup_{0<\varepsilon<<1}|u(x_i)-U_i| \leq \frac{CN^{-2}}{N^{-1}+\varepsilon} \leq CN^{-1} \tag{4.47}$$

Proof By substituting the results of lemma 5 into Theorem 1 and applying the discrete maximum principle, we obtain the required bound.

For the case $\varepsilon > N^{-1} = h$, the scheme secures second-order convergence and we expect to lose an order of convergence for $\varepsilon \leq N^{-1} = h$. In fact, it turns out that the scheme guarantees first-order uniformly convergent.

4.5 Numerical Example and Results

In this section, one example is given to illustrate the numerical method discussed above. The exact solutions to the test problem are not known. So that we can apply the Double Mesh Principle.

Definition 4.5:-The Double-mesh principle is used to calculate the maximum absolute error and rate of convergence of the numerical scheme, when the differential equation has no exact solutions[3] . So the accuracy of their numerical solutions will be computed using the double mesh principle. Therefore, we use the double mesh principle to estimate the error and compute the experimental rate of convergence to the computed solution. For any value of N,we put

$$E_\varepsilon^N = \max_{0 \le i \le 2N} \left| U_i^N - U_{2i}^{2N} \right|,$$

where U_i^N and U_{2i}^{2N} are the i-th and $2i$-th components of the numerical solutions on meshes of N and $2N$ respectively. We compute the uniform error and the rate of convergence as:

$$E^N = \max_{\varepsilon,\mu} E_\varepsilon^N$$

and

$$R^N = \log_2 \left(\frac{E^N}{E^{2N}} \right)$$

Example 4.1

Consider the boundary value problem for the delay differential equation:

$$\begin{cases} \varepsilon u''(x) + \mu(1+x)u'(x) - (4+\sin x)u(x) + u(x-1) = -e^x, \quad x \in (0,2), \\ u(x) = 5 + x, \text{ for } x \in [-1,0], \\ u(2) = 5. \end{cases}$$

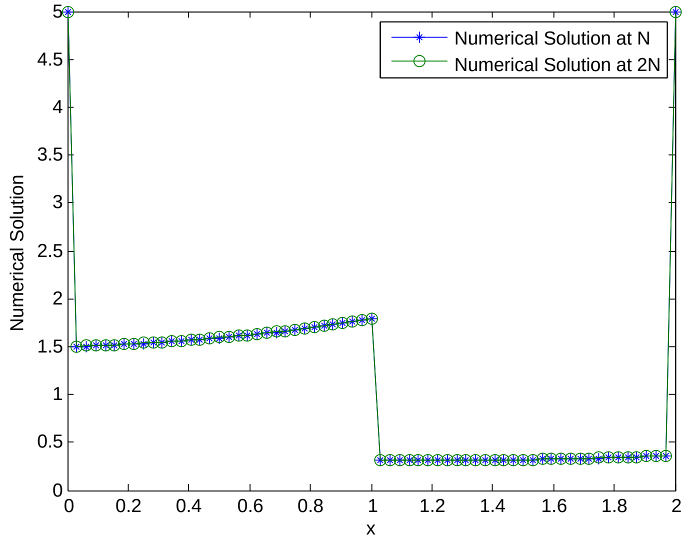

Figure 4.1: The behavior of the Numerical Solution for Example 4.1 at $\varepsilon = 2^{-12}$,$\mu = 2^{-22}$,and $N = 64$

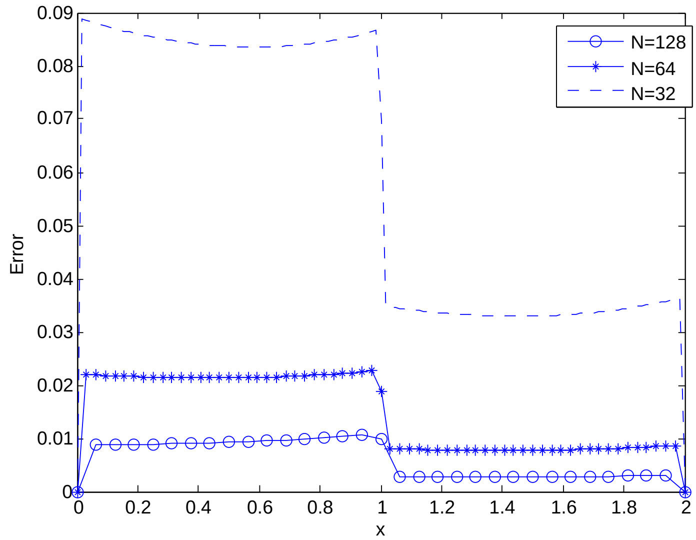

Figure 4.2: Point wise absolute error of the given Example 4.1 Solution at $\varepsilon = 2^{-43}$,$\mu = 2^{-15}$ $N = 32, 64 \quad and \quad 128$

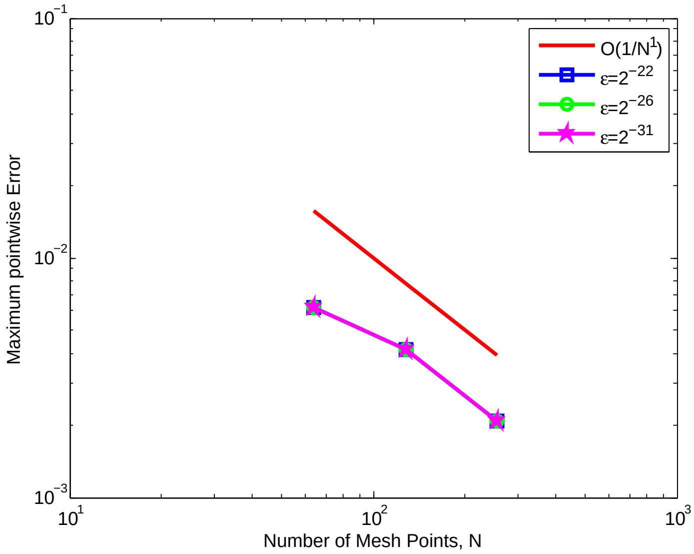

Figure 4.3: ε-uniform convergence with the fitted operator log-log scale the given Example 4.1 of a constant μ and different values of ε=$2^{-22}, 2^{-26}, 2^{-31}$.

Table 4.1: Maximum absolute error and rate of convergence for different values of ε,μ, and N with exponential fitted operator method for the given Example 4.1

$\varepsilon \in \{\frac{1}{2^{11+k}}, 1 \leq k \leq 10\}$

	Present method			Kalaiselvan *et al.*(2019)		
	Number of mesh elements N					
μ	64	128	256	64	128	256
2^{-22}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-23}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-24}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-25}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-26}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-27}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-28}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-29}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-30}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
2^{-31}	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
E^N	0.61892E−02	0.41362E−02	0.20865E−02	0.4756E−01	0.3276E−01	0.1486E−01
R^N	0.58145E+00	0.98722E+00		0.53773E+00	0.1140E+01	

Table 4.2: Comparison of maximum absolute error and rate of convergence for different values of ε, μ, and N with the exponential fitted operator method and fitted mesh method for the given Example 4.1

$\mu \in \{\frac{1}{2^{5+k}}, 1 \leq k \leq 10\}$

	Present method			Kalaiselvan *et al.*(2019)		
	Number of mesh elements N					
ε	128	256	512	128	256	512
2^{-34}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-35}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-36}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-37}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-38}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-39}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-40}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-41}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-42}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
2^{-43}	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
E^N	0.29621E−01	0.58327E−02	0.18448E−02	0.1241E+00	0.9759E−01	0.6750E−01
R^N	0.234438E+01	0.16607E+01		0.3471E+00	0.5319E+00	

Chapter 5

Conclusion and Future work

5.1 Conclusion

This study introduces an exponentially fitted numerical method for singularly perturbed differential equation with two parameters with large delay. The numerical scheme is developed on uniform mesh using finite difference approximations operator in the given differential equation. The stability of the developed numerical method is established its uniform convergence is proved. To validate the applicability of the method, one model problem is considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors and the numerical rate of convergence, and uniform errors. The performance of the proposed scheme is investigated by comparing it with the beforehand study. The proposed method is more accurate, stable, and uniform convergent numerical result.

5.2 The Future Work

In this thesis, exponentially fitted numerical methods were constructed for solving singularly perturbed differential equations. Hence, the scheme proposed in this thesis can also be extended to solve singularly perturbed differential problems with two small parameters and large delays.

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