Fixed point results of generalized Suzuki-type contraction mapping in quasi b-metric spaces

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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Declaration

I, undersigned declare that, this research paper entitled “fixed Point results of generalized Suzuki type-contraction mapping in a quasi b-metric spaces” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this research we established fixed point theorem for generalized Suzuki-type contraction mapping in quasi b-metric space and proved the existence and uniqueness of fixed point for self-mappings satisfying the established theorem. Our result generalized and extend the work of Ali et al., (2023) from quasi-metric space. Finally, we provided examples in support of our main finding. In this research under taking, we followed analytical study design and used secondary sources of data such as published paper related books.

Contents

Chapter 1

Introduction

1.1 Background of the study

Fixed point theory is an important tool in the study of nonlinear analysis. It is considered to be the key connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics and almost all engineering fields. In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the "Banach contraction principle (BCP)" which is one of the most important results of nonlinear analysis and considered as the main source of metric fixed point theory. It confirms the existence and uniqueness of fixed point of self-maps of metric spac and provides a constructive method to find fixed point. Another category of contraction which is separate from Banach contraction and does not imply continuity, was proposed by Kannan, (1968) who also established in the same work that such mappings necessarily have unique fixed points in complete metric spaces. Mappings belonging to this category are known as Kannan type. In 1972, a new concept which is different from that of Bannach, (1922) and Kannan, (1968) for contraction type mapping was introduced by Chatterjea, (1972) which gives a new direction to the study of fixed-point theory. There are classes of contractive mappings which are different from Banach contraction and have unique fixed point in complete metric space. Suzuki (2008) and Kikkawa and Suzuki (2008) generalized Banach contraction principle by introducing Suzuki type contractions of metric space. Note that Suzuki-type contractions characterize the completeness of underlying metric spaces. Moreover, Suzuki-type contractions are in the form of implication which allows many discontinuous mappings of metric spaces to be Suzuki-type contractions, while Banach contractions are necessarily continuous mappings of metric spaces. Suzuki-type contractions have a wide scope of applications, for instance, in delay differential equations. Romaguera (2022) discussed fixed point results for generalized Ciric's contractions of quasi-metric spaces. Romaguera's investigations revealed that unlike the Banach contraction principle and Kannan-type fixed point theorem, a basic Suzuki-type fixed point theorem cannot be transported as it is in the setup of quasi-metric spaces. Romaguera proposed a new type of Suzuki-type contraction by tweaking the original Suzuki type- contraction a little bit and obtained a variant of Suzuki type fixed point theorems for mappings of quasi-metric spaces. Romaguera introduced a 2-basic contraction of the Suzuki type and proved fixed point results in the setup of Smyth's complete quasi-metric space. Note that 2-basic contraction of the Suzuki type does not reduce to the original Suzuki type contraction whenever quasi-metric is replaced by metric. Ali et al., (2023) proposed a new type of Suzuki-type contraction, which is reduced to the original Suzuki-type contraction when quasi-metric is replaced with metric. Suzuki-type contractions cannot be transported in quasi-metric spaces, so the transportation of generalization of such contractions is out of the question. There is a special type of quasi-metric space known as D-symmetric quasi-metric spaces. These spaces are Hausdorff quasi-metric spaces. Ali et al., (2023) also proved the existence and uniqueness of fixed points of generalized Suzuki-type contractions of D-symmetric quasi-metric spaces. Inspired and motivated by the work of Ali et al., (2023) the main purpose of this research work is to define generalized Suzuki-type contraction mapping in a quasi b-metric spaces, establish a fixed-point result and prove the existence and uniqueness of a fixed points for the class of Suzuki-type Contraction mappings introduced in the setting of a quasi b-metric Spaces.

1.2 Statement of the Problem

Romaguera, (2022) proposed a new type of Suzuki-type contraction by tweaking the original Suzuki type contraction a little bit and obtained a variant of Suzuki type fixed point theorems for mappings of quasi-metric spaces. Romaguera introduced a 2-basic contraction of the Suzuki type and proved fixed point results in the setup of Smyth's complete quasi-metric spaces. In 2023, Ali et al., (2023) introduced the notions of Suzuki typecontraction and studied existence and uniqueness of fixed point results in the setting of quasimetric spaces. However, a fixed point results of generalized Suzuki type-contraction mapping in the setting of quasi b-metric space is not yet studied. Inspired and motivated by the research findings of the above researchers, in this research work we concentrated in defined generalized Suzuki type-contraction mappings in the setting of quasi b-metric spaces and study a fixed-point result for the mapping introduced in the setting of quasi b-metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this thesis is to study fixed point results of generalized Suzuki-type contraction mapping in the setting of quasi b-metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives

- To define generalized Suzuki-type contraction mapping in the setting of quasi b-metric spaces.
- To establish fixed point theorems for generalized Suzuki-type contraction mappings in quasi b-metric spaces.
- To prove existence and uniqueness fixed points results of generalized Suzuki-type contractions in the setting of quasi b-metric spaces.
- To provide examples in support of the main result obtained.

1.4 Significance of the study

The result of this study may have the following importance

- The outcome of this study may contribute to research activities in the study area.
- It may provide basic research skills to the researcher.
- It may help to show existence and uniqueness of solutions of some problems involving integral and differential equation.

1.5 Delimitation of the Study

This study was delimited to studying a fixed-point result for generalized Suzuki-type contraction mapping in the setting of quasi b-metric spaces.

Chapter 2

Review of Related literatures

Fixed point theory is one of the most dynamic research subjects in nonlinear analysis. Let X be anonempty set and $T : X \to X$ be a self mapping. We say that x is a fixed point of T if $T(x) = x$ and denoted by $Fix(T)$. Fixed point theory has been studied extensively, which can be seen from the works of many authors.

Theorem 2.1 *(Banach, 1922) Let (X,d) be a complete metric space and T be a contraction on X, i.e., there exists $r \in [0,1)$ such that $d(Tx,Ty) \leq rd(x,y)$ for all $x,y \in X$. Then T has a unique fixed point.*

Theorem 2.2 *(Suzuki, 2008) Let (X,d) be a complete metric space, and T be a mapping on X. Define a nonincreasing function θ from $[0,1)$ into $(1/2,1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r \leq 1,. \end{cases}$$

Assume that there exists $r \in [0,1)$, such that $\theta(r)d(x,Tx) \leq d(x,y)$ implies $d(Tx,Ty) \leq rd(x,y)$, for all $x,y \in X$, then there exists a unique fixed-point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Theorem 2.3 *(Hussain et al., 2015) Let (X,d) be a complete metric space, and T be a mapping on X. Define a non-increasing function θ from $[0,1)$ into $(1/2,1]$ as in the (Suzuki, 2008) Assume that there exists $F \in \psi$, such that $\theta(r)d(x,Tx) \leq d(x,y)$ implies $F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),dy,Tx)) \leq 0$, for all $x,y \in X$, then T has a unique fixed-point z and $l\lim_{n\to\infty} T^n x = z$ for every $x \in X$.*

Theorem 2.4 *(Romaguera, 2022) Let T be a 2-basic contraction of Suzuki-type on a Smyth complete quasi-metric space (X,d). Then, for each $x_0 \in X$ there exists $y \in X$ such that $d(Ty,y) = 0$ and $d\max(y,T^n x_0) \to 0$ as $n \to \infty$.*

Theorem 2.5 *(Kannan, 1968) Let (X,d) be a metric space and $T : X \to X$. Then T is called a Kannan mapping if there exists $k \in [0,\frac{1}{2})$ such that*

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)] \text{ for all } x,y \in X.$$

If (X,d) is a complete metric space, then T has a unique fixed point.

Definition 2.1 *(Rus, 2008) Let $\psi : [0,\infty) \to [0,\infty)$ be a mapping. Then ψ is called comparison function if it is satisfying the following conditions*
(i) ψ is increasing;
(ii) $\psi(t) < t$, for all $t > 0$;
(iii) $\psi(0) = 0$;
(iv) $\psi^n(t) \to 0$ as $n \to \infty$ for all $t \in [0,\infty)$;
(v) $t - \psi(t) \to \infty$ as $t \to \infty$;
(vi) $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for all $t \in [0,\infty)$.

Definition 2.2 *(Shah and Hussain, 2012) Let X be a nonempty set. A real-valued function $d;X \times X \to [0,\infty)$ is said to be a quasi b-metric on X with the constant $s \geq 1$ if the following conditions are satisfied:*
(d_1) $d(x,y) \geq 0$ for all $x,y \in X$;
(d_2) $d(x,y) = 0$ if and only if $x = y$;
(d_3) $d(x,z) \leq s(d(x,y) + d(y,z))$for all $x,y,z \in X$.
The pair (X,d) is called a quasi b-metric space.

Observe that if $s = 1$, then the ordinary triangle inequality of quasi-metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of quasi b-metric spaces is effectively larger than that of the ordinary quasi-metric space. That is, every quasi-metric space is a quasi b-metric space but the converse need not be true.

Definition 2.3 *(Ali et al., 2023) For a quasi-metric space (X,d) and a mapping $f : X \to X$. If there is $\alpha \in [0,1)$ such that $\lambda(\alpha)d(x, f(x)) \leq d(x,y) \Rightarrow d(f(x), f(y)) \leq \alpha d(x,y)$ for all $x, y \in X$. Then f is called Suzuki type-contraction on (X,d).*

Theorem 2.6 *(Ali, et al., 2023) Let (X,d) be a d-sequentially complete Δ-symmetric quasi-metric spaces. Then every Generalized Suzuki-type contraction on (X,d) has a unique fixed point.*

Inspired and motivated by the work of Ali et al., (2023) the main purpose of this research work is to define generalized Suzuki-type contraction mapping in a quasi b-metric spaces, establish a fixed point results and prove the existence and uniqueness of a fixed points for the class of Suzuki-type Contraction mappings introduced in the setting of a quasi b-metric Spaces.

Chapter 3

Methodology

3.1 Study period and site

The study was conducted at Jimma University department of mathematics under the Functional Analysis stream from September, 2023 G.C to June, 2024 G.C.

3.2 Study Design

In this research work we employed analytical design.

3.3 Source of Information

The relevant sources of information for this study were books, published articles and related studies from internet.

3.4 Mathematical Procedure of the Study

In this research under taking, we followed the standard procedures. The procedures are:

1. Defining generalized Suzuki-type contraction mapping in quasi b-metric Spaces.
2. Establishing a fixed-point theorem for generalized Suzuki-type contraction mapping in quasi b-metric Spaces.

3. Constructing a sequence in quasi b-metric spaces.
4. Showing the constructed sequence is Cauchy in quasi b-metric Spaces.
5. Showing the convergence of the sequence in quasi b-metric spaces.
6. Proving the existence of a fixed point result of generalized Suzuki-type contraction mapping in quasi b-metric Spaces.
7. Showing uniqueness of the fixed point result of generalized Suzuki-type contraction mapping in quasi b-metric Spaces.
8. Providing an example in support of our main result.

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Now, we state some notation, definitions and theorems which are useful for our work as follows.

Notation Through out this work we denotes:

(i) ${\mathbb{R}_0}^+ = [0,\infty)$;

(ii) $\mathbb{N}$ is the set of natural numbers;

(iii) $\mathbb{R}^+$ is the set of all positive real numbers;

(iv) $\mathbb{R}$ is the set of all real numbers;

(iv) λ denote decreasing function.

Definition 4.1 *(Czerwik, 1993) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to {\mathbb{R}_0}^+$ is a b-metric on X if, for all, $x,y,z \in X$, the following conditions hold:*

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$ (b-triangular inequality).

Then the pair (X,d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces; every metric is a b-metric with s = 1, while the converse is not true.

Definition 4.2 *(Shah and Hussain, 2012)) Let X be a nonempty set. A function $d : X \times X \to {\mathbb{R}_0}^+$ is said to be a quasi b- metric on X with the constant $s \geq 1$ if the*

following Conditions are satisfied:
(i) $d(x,y) \geq 0$ *for all* $x, y \in X$;
(ii) $d(x,y) = 0$ *if and only if* $x = y$;
(iii) $d(x,z) \leq s(d(x,y) + d(y,z))$ *for all* $x, y, z \in X$.
The pair (X,d) *is called a quasi b-metric space.*

Observe that if $s = 1$, then the triangle inequality of ordinary quasi-metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of quasi b-metric space is effectively larger than that of the ordinary quasi-metric space. That is, every quasi-metric space is a quasi b-metric space but the converse need not be true.
The following example explains the above-mentioned situation.

Example 4.1 *Let* $X = C([0,1], \mathbb{R})$ *the class of all continuous real valued function defined on* $[0,1]$. *Define* $d : X \times X \to \mathbb{R}_0{}^+$ *by*

$$d(f,g) = \begin{cases} \int\limits_0^1 [g(t) - f(t)]^3 dt & \text{if } f \leq g\}, \\ \int\limits_0^1 [f(t) - g(t)]^3 dt & \text{if } f \geq g\},. \end{cases}$$

Note that $d(f,g) \geq 0$ *for all* $f, g \in X$ *and* $d(f,g) = 0$ *if and only if* $f = g$. *Also,* $d(f,g) = d(g,f)$ *if and only if* $f = g$ *so that* d *is not symmetric. Let* $f(t) = 2t$, $g(t) = 5t$ *and* $h(t) = 6t$ *for* $t \in [0,1]$. *Then* $d(f,h) = 16$;
$d(f,g) = 27/4$;
$d(g,h) = 1/4$.
That is, $d(f,h) > d(f,g) + d(g,h)$ *so that the usual triangle inequality quasi-metric space is not satisfied. Suppose that there exists* $s \geq 1$ *such that* $d(f,h) \leq s[d(f,g) + d(g,h)]$ *then putting the values and simplifying we get* $16 \geq s[27/4 + 1/4] = 16 \leq 7s = s \geq (16)/7$.
Thus, for every $f, g.h \in X$, *whenever the usual quasi-metric triangle inequality fails to hold. We can find an* $s \geq 1$ *such that the triangle inequality of the quasi b-metric space is satisfied.*

(Aphane and Moshokoa, (2015) Given a quasi b-metric space (X,d) defined d^{-1} :

$X \times X \to \mathbb{R}_0{}^+$ by $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$. Then (X, d^{-1}) is also a quasi b-metric space. We call d^{-1} is the conjugate of d on X.
Next define the function $d^* : X \times X \to \mathbb{R}_0{}^+$ by $d^*(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$.
Note that d^* is b-metric on X.

Definition 4.3 *(Shah and Hussain, 2012) Let (X,d) be a quasi b-metric space and $\{x_n\}$ be a sequence in X. Then*
(a) The sequence $\{x_n\}$ converges to x in X if and only if $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x_n,x) = 0$;
(b) The sequence $\{x_n\}$ is left Cauchy if and only if for every $\varepsilon > 0$ there exists positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$for all $n \geq m > N$;
(c) The sequence $\{x_n\}$ is right Cauchy if and only if for every $\varepsilon > 0$ there exists positive positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$for all $m \geq n > N$;
(d) The sequence $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists positive positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $n,m > N$.

Note
(i) A sequence $\{x_n\}$ in a quasi b-metric space is Cauchy if it is left Cauchy and right Cauchy.
(ii) Let (X,d) be quasi b-metric space. We say that (X,d) is complete if and only if each Cauchy sequences in X is converges to a point $x \in X$.

4.2 Main Results

In this section we introduced generalized Suzuki type contraction mapping in the setting of quasi b-metric spaces and proved fixed point result for the mapping introduced. Throughout this thesis, λ denotes a decreasing function $\lambda : [0, \frac{1}{s}) \to (\frac{1}{2s}, \frac{1}{s}]$ defined as $\lambda(\alpha) = \frac{1}{s+\alpha s^2}$ for all $\alpha \in [0, \frac{1}{s})$ and $s \geq 1$.

Definition 4.4 *For a quasi b-metric space (X,d) with $s \geq 1$ and a mapping $T : X \to X$.*
a) If there is $\alpha \in [0, \frac{1}{s})$ such that $d(Tx,Ty) \leq s\alpha d(x,y)$;
$d(Tx,Ty) \leq s\alpha d(y,x)$, for all $x,y \in X$.
Then T is called d-contraction and $d_{\{}-1\}$-contraction on (X,d) respectively.
(b) If there is $\alpha \in [0, \frac{1}{s})$ such that
$\lambda(\alpha)d(x,Tx) \leq d^*(x,y) \Rightarrow d^*(Tx,Ty) \leq s\alpha d(x,y)$, for all $x,y \in X$.
Then T is called Suzuki type d-contraction on (X,d).
(c) If there is $\alpha \in [0, \frac{1}{s})$ such that
$\lambda(\alpha)d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq s\alpha d(x,y)$, for all $x,y \in X$.
Then T is called Suzuki type–contraction mapping on quasi b-metric spaces (X,d).
(d) If there is $\alpha \in [0, \frac{1}{s})$ such that
$\min d(x,Tx), d(y,Ty \leq 2d(x,y) \Rightarrow d(Tx,Ty) \leq s\alpha d(x,y)$ *for all $x,y \in X$.*
Then T is called 2-basic-contraction mapping of Suzuki type on quasi b-metric spaces (X,d).

Definition 4.5 *For a positive real number $\Delta \geq 0$, a quasi b-metric space (X,d) is said to be Δ-symmetric if $d(x,y) \leq \Delta d(y,x)$, for all $x,y \in X$.*

Note: Δ-symmetric quasi b-metric space (X,d) constitutes a b-metric space if $\Delta = 1$.

Definition 4.6 *A quasi b-metric space (X,d) is called d-sequentially complete if every Cauchy sequence in the b-metric space (X,d^*) converges with respect to the topology τ_d on X.*

Note: A quasi b-metric space (X,d) is bicomplete if the b-metric space (X,d^*) is complete.

Definition 4.7 *Let (X,d) be quasi b-metric spaces with $s \geq 1$ and $T : X \to X$ a mapping. If there is $\alpha \in [0, \frac{1}{s})$, for all $x, y \in X$ such that*

$$\lambda(\alpha)d(x,Tx) \leq d^*(x,y) \Rightarrow d(Tx,Ty) \leq s\alpha U_T(x,y), \tag{4.1}$$

where

$$U_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(Tx,T^2x), \frac{d(T^2x,Ty)}{2s}, \frac{(d(y,Tx)+d(x,Ty))}{2s}\}.$$

Then T is called generalized Suzuki type–contraction mapping on quasi b-metric space (X,d).

Theorem 4.1 *Let (X,d) be d-sequentially complete Δ-symmetric quasi b-metric spaces, then every generalized Suzuki type–contraction mapping on (X,d) has unique fixed point.*

Proof: Since by Definition 4.7 T is generalized Suzuki type –contraction on (X,d) and α be real number such that $0 \leq \alpha < \frac{1}{s}$ for a fixed $x \in X$ we defined an iterative sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If there exist an $n \in \mathbb{N} \cup \{0\}$ such that $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point of T which completes the proof. Now, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\lambda(\alpha) \leq \frac{1}{s}$, we have $\lambda(\alpha)d(x_n,Tx_n) \leq d(x_n,Tx_n) = d(x_n,x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. By the given hypothesis also

$$d(x_{n+1},x_{n+2}) = d(Tx_n,Tx_{n+1}) \leq s\alpha U_T(x,y), \tag{4.2}$$

where

$$\begin{aligned} U_T(x,y) \quad = \quad \max\Bigg\{ & d(x_n,x_{n+1}), d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}), d(Tx_n,T^2x_n), \\ & \frac{d(T^2x_n,Tx_{n+1})}{2s}, \frac{(d(x_{n+1},Tx_n)+d(x_n,Tx_{n+1}))}{2s} \Bigg\}. \end{aligned}$$

Thus, from (4.2), we have

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &\leq s\alpha \max\Bigg\{ d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), d(Tx_n, T^2x_n), \\
&\quad \frac{d(T^2x_n, Tx_{n+1})}{2s}, \frac{(d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1}))}{2s} \Bigg\} \\
&= s\alpha \max\Bigg\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2}), \\
&\quad \frac{d(x_{n+2}, x_{n+2})}{2s}, \frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2}))}{2s} \Bigg\} \\
&= s\alpha \max\Bigg\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}))}{2s} \Bigg\} \\
&\leq \max\Bigg\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s} \Bigg\} \\
&= s\alpha \max\Bigg\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \Bigg\},
\end{aligned}$$

that is, $d(x_{n+1}, x_{n+2}) \leq s\alpha \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$.
If $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$, then from the above inequality, we get

$$d(x_{n+1}, x_{n+2}) \leq s\alpha d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n+2}),$$

which is a contradiction. Therefore, we obtain $d(x_{n+1},x_{n+2}) \leq s\alpha d(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$. By continuing in this process again from (4.2), we get

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\
&\leq s\alpha d(x_{n-1},x_n) \\
&= s\alpha d(Tx_{n-2},Tx_{n-1}) \\
&\leq s^2\alpha^2 d(x_{n-2},x_{n-1}) \\
&\cdot \\
&\cdot \\
&\cdot \\
&\leq s^n\alpha^n d(x_0,x_1) \\
&= \alpha^n(s^n d(x_0,x_1)) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{4.3}$$

Similarly,

$$\begin{aligned}
d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\
&\leq s\alpha d(x_n,x_{n-1}) \\
&= s\alpha d(Tx_{n-1},Tx_{n-2}) \\
&\leq s^2\alpha^2 d(x_{n-1},x_{n-2}) \\
&\cdot \\
&\cdot \\
&\cdot \\
&\leq s^n\alpha^n d(x_1,x_0) \\
&= \alpha^n(s^n d(x_1,x_0)) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0. \tag{4.4}$$

Therefore,

$$\lim_{n\to\infty} \max\{d(x_n,x_{n+1}),d(x_{n+1},x_n)\} = \lim_{n\to\infty} d^*(x_n,x_{n+1}) = 0.$$

Now we prove that $\{x_n\}$ is Cauchy sequence in quasi b-metric space (X,d). First, we want to show that $\{x_n\}$ is right Cauchy sequence in quasi b-metric space (X,d). Using the triangle inequality of quasi b-metric space, for $n,k \in \mathbb{N}$ and $s \geq 1$, we get

$$\begin{aligned}
d(x_n,x_{n+k}) &\leq s(d(x_n,x_{n+1})+d(x_{n+1},x_{n+k})) \\
&\leq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+s^3d(x_{n+1},x_{n+1})+\ldots+s^kd(x_{n+k-1},x_{n+k}) \\
&\leq s\alpha^n d(x_0,x_1)+s^2\alpha^{n+1}d(x_0,x_1)+s^3\alpha^{n+2}d(x_0,x_1)+\ldots+s^k\alpha^{n+k-1}d(x_0,x_1) \\
&\leq s\alpha^n[1+s\alpha+(s\alpha)^2+(s\alpha)^3+\ldots+s^{k-1}\alpha^{k-1}]d(x_0,x_1) \\
&\leq \frac{s\alpha^n}{1-s\alpha}d(x_0,x_1) \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence, from the above inequality, we obtain

$$\lim_{n\to\infty} d(x_n,x_{n+k}) = 0, \text{ for all } k, \tag{4.5}$$

that is, $\{x_n\}$ is a right Cauchy sequence in the quasi b-metric spaces (X,d).
Similarly, we show that $\{x_n\}$ is a left Cauchy sequence in quasi b-metric spaces as

follows.

$$\begin{aligned}
d(x_{n+k},x_n) &\leq s(d(x_{n+k},x_{n+k-1})+d(x_{n+k-1},x_n))\\
&\leq sd(x_{n+k},x_{n+k-1})+s^2d(x_{n+k-1},x_{n+k-2})+s^3d(x_{n+k-2},x_{n+k-3})\\
&\quad +\ldots+s^kd(x_{n+1},x_n)\\
&\leq s\alpha^n d(x_1,x_0)+s^2\alpha^{n+1}d(x_1,x_0)+s^3\alpha^{n+2}d(x_1,x_0)\\
&\quad +\ldots+s^k\alpha^{n+k-1}d(x_1,x_0)\\
&\leq s\alpha^n[1+s\alpha+(s\alpha)^2+(s\alpha)^3+\ldots+s^{k-1}\alpha^{k-1}]d(x_1,x_0)\\
&\leq \frac{s\alpha^n}{1-s\alpha}d(x_1,x_0)\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence, from the above inequality, we obtain

$$\lim_{n\to\infty} d(x_{n+k},x_n)=0, \text{ for all } k, \tag{4.6}$$

that is, $\{x_n\}$ is a left Cauchy sequence in the quasi b-metric spaces (X,d). Thus, from (4.5) and (4.6), we conclude that

$$\lim_{n\to\infty} d(x_n,x_{n+k})=0=\lim_{n\to\infty} d(x_{n+k},x_n).$$

Hence, $\{x_n\}$ is a Cauchy sequence in the quasi b-metric spaces (X,d). Since (X,d) is d-sequential complete (Cauchy sequence in b-metric space (X,d^*) converges with respect to the topology τ_d), there exists $x_0\in X$ such that

$$\lim_{n\to\infty} d(x_n,x_0)=0. \tag{4.7}$$

Now we claim that

$$d(x_0,Tx)\leq \alpha s\max\{d(x_0,x),d(x,x_0),d(x,Tx\}, \text{ for all } x\neq x_0. \tag{4.8}$$

If $\max\{d(x_0,x),d(x,x_0),d(x,Tx)\} = d(x,x_0)$, then $d(x_0,Tx) \leq \alpha s d(x,x_0) < d(x,x_0)$. By (4.8) there exists an $n_0 \in \mathbb{N}$ such that

$$\Delta d(x_0,x_n) < \frac{1}{3}d(x_0,x) = d(x_0,x_n) < \frac{1}{3\Delta}d(x_0,x). \tag{4.9}$$

Using the triangle inequality of quasi b-metric space, Δ-symmetricproperty and (4.9) for all $n \geq n_0$, we get

$$\begin{aligned}
\lambda(\alpha)d(x_n,Tx_n) &\leq d(x_n,Tx_n) \\
&= d(x_n,x_{n+1}) \\
&\leq s(d(x_n,x_0)+d(x_0,x_{n+1})) \\
&\leq s[\Delta d(x_0,x_n)+d(x_0,x_{n+1})] \\
&\leq s[\Delta(\frac{1}{3\Delta}d(x_0,x)+\frac{1}{3\Delta}d(x_0,x))] \\
&= \frac{2s}{3}d(x_0,x) \\
&= s[d(x_0,x)-\frac{1}{3}d(x_0,x)] \\
&\leq s[d(x_0,x)-\Delta d(x_0,x_n)] \\
&\leq s[d(x_0,x)-d(x_0,x_n)] \\
&\leq s[\max\{d(x_n,x),d(x,x_n)\}] \text{ for any } s \geq 1 \\
&= d(x_0,x)-d(x_0,x_n) \\
&\leq \max\{d(x_n,x),d(x,x_n)\}.
\end{aligned}$$

Thus, $\lambda(\alpha)d(x_n,Tx_n) \leq d^*(x_n,x)$ holds for all $n \geq n_0$. From (4.1), we have

$$d(x_{n+1},Tx) = d(Tx_n,Tx) \leq s\alpha U_T(x_n,x),$$

where

$$U_T(x_n,x) = \max\{d(x_n,x),d(x_n,Tx_n),d(x,Tx),d(Tx_n,T^2x_n),\frac{d(T^2x_n,Tx)}{2s},\frac{(d(x,Tx_n)+d(x_n,Tx))}{2s})\}.$$

Thus

$$\begin{aligned} d(x_{n+1},Tx) &= d(Tx_n,Tx) \\ &\leq s\alpha\max\Bigg\{d(x_n,x),d(x_n,Tx_n),d(x,Tx),d(Tx_n,T^2x_n), \\ &\quad \frac{d(T^2x_n,Tx)}{2s},\frac{(d(x,Tx_n)+d(x_n,Tx))}{2s})\Bigg\} \\ &= s\alpha\max\Bigg\{d(x_n,x),d(x_n,x_{n+1}),d(x,Tx),d(x_{n+1},x_{n+2}), \\ &\quad \frac{d(x_{n+2},Tx)}{2s},\frac{(d(x,x_{n+1})+d(x_n,Tx))}{2s}\Bigg\}. \end{aligned}$$

Taking limit as $n\to\infty$ in the above inequality, we get

$$\begin{aligned} d(x_0,Tx) &\leq s\alpha\max\Bigg\{d(x_0,x),d(x,Tx),\frac{d(x_0,Tx)}{2s},\frac{(d(x,x_0)+d(x_0,Tx))}{2s}\Bigg\} \\ &\leq s\alpha\max\Bigg\{d(x_0,x),d(x,Tx),\frac{(s[d(x_0,x)+d(x,Tx)])}{2s},\frac{(s[d(x,x_0)+d(x_0,Tx)])}{2s}\Bigg\} \\ &= s\alpha\max\Bigg\{d(x_0,x),d(x,Tx),\frac{([d(x_0,x)+d(x,Tx)])}{2},\frac{(d(x,x_0)+d(x_0,Tx))}{2}\Bigg\} \\ &\leq s\alpha\max\Bigg\{d(x_0,x),d(x,Tx),\frac{(d(x,x_0)+d(x_0,Tx))}{2}\Bigg\}. \end{aligned}$$

If $\max\{d(x_0,x),d(x,Tx),\frac{(d(x,x_0)+d(x_0,Tx))}{2}\}=\frac{(d(x,x_0)+d(x_0,Tx))}{2}$.

Then

$$
\begin{aligned}
d(x_0,Tx) &\leq s\alpha\frac{(d(x,x_0)+d(x_0,Tx))}{2}\\
&= 2d(x_0,Tx)\\
&\leq s\alpha d(x,x_0)+s\alpha d(x_0,Tx)\\
&= 2(x_0,Tx)-s\alpha d(x_0,Tx)\\
&\leq s\alpha d(x,x_0)\\
&= (2-s\alpha)(x_0,Tx)\\
&\leq s\alpha d(x,x_0).
\end{aligned}
$$

Thus, we have

$$
d(x_0,Tx)\leq \frac{s\alpha}{(2-s\alpha)}d(x,x_0)<s\alpha d(x,x_0)\leq s\alpha\max\{d(x_0,x),d(x,x_0),d(x,Tx)\}.
$$

Hence, (4.8) holds, for all $x\neq x_0$.
Now, we prove $x_0=Tx_0$. Assume on contrary that

$$
x_0\neq Tx_0,\ \text{for all } x_0\in X. \tag{4.10}
$$

Set $x=Tx_0$ in view of (4.10), $x\neq x_0$. Since $\lambda(\alpha)d(x_0,Tx_0)\neq d(x_0,Tx_0)=d(x_0,x)$.

Thus,

$$
\lambda(\alpha)d(x_0,Tx_0)\leq d^*(x_0,x).
$$

By (4.1), we have

$$
\lambda(\alpha)d(x,Tx)\leq d(x,Tx)=d(Tx_0,Tx)\leq s\alpha U_T(x_0,x),
$$

where

$$
\begin{aligned}
U_T(x_0,x) &= \max\Big\{d(x_0,x),d(x_0,Tx_0),d(x,Tx),d(Tx_0,T^2x_0),\\
&\qquad \frac{d(T^2x_0,Tx)}{2s},\frac{(d(x,Tx_0)+d(x_0,Tx))}{2s}\Big\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d(x,Tx) &= d(Tx_0,Tx) \\
&\leq s\alpha \max\Bigg\{(x_0,x), d(x_0,Tx_0), d(x,Tx), d(Tx_0,T^2x_0), \\
&\quad \frac{d(T^2x_0,Tx)}{2s}, \frac{(d(x,Tx_0)+d(x_0,Tx))}{2s}\Bigg\} \\
&\leq s\alpha \max\left\{d(x_0,x), d(x,Tx), \frac{d(x,Tx)}{2s}, \frac{(s[d(x_0,x)+d(x,Tx)])}{2s}\right\} \\
&\leq s\alpha \max\{d(x_0,x), d(x,Tx)\}.
\end{aligned}
$$

If $\max\{d(x_0,x), d(x,Tx)\} = d(x,Tx)$, then $d(x,Tx) \leq s\alpha d(x,Tx) < d(x,Tx)$, which is a contradiction. Hence, we get $d(x,Tx) \leq s\alpha d(x_0,x)$. Thus, along with (4.8), it implies that

$$
d(x_0,Tx) \leq s\alpha \max\{d(x_0,x), d(x,x_0)\}. \tag{4.11}
$$

Now applying the triangle inequality of quasi b-metric spaces, we have

$$
\begin{aligned}
d(x,Tx) &\leq s[d(x,x_0)+(x_0,Tx)]) \\
&\leq s[d(x,x_0)+s\alpha \max\{d(x_0,x), d(x,x_0)\}] by (4.11) \\
&\leq sd^*(x,x_0)+s^2\alpha d^*(x,x_0) \\
&\leq (s+s^2\alpha)d^*(x,x_0).
\end{aligned}
$$

Thus,

$$
\frac{1}{(s+s^2\alpha)} d(x,Tx) \leq d^*(x,x_0) = \lambda(\alpha) d(x,Tx) \leq d^*(x,x_0).
$$

Using, (4.1), we have

$$
d(Tx,Tx_0) \leq s\alpha U_T(x,x_0),
$$

where

$$U_T(x,x_0)=\max\left\{d(x,x_0),d(x,Tx),d(x_0,Tx_0),d(Tx,T^2x),\frac{d(T^2x,Tx_0)}{2s},\frac{(d(x_0,Tx)+d(x,Tx_0))}{2s}\right\}.$$

Thus,

$$d(Tx,Tx_0)\leq s\alpha\max\left\{d(x,x_0),d(x,Tx),d(x_0,Tx_0)d(Tx,T^2x),\frac{d(T^2x,Tx_0)}{2s},\frac{(d(x_0,Tx)+d(x,Tx_0))}{2s}\right\}$$

holds for all $x\neq x_0\in X$.
Putting $x=x_n$ in the above inequality, we get

$$\begin{aligned}
d(x_{n+1},Tx_0) &= d(Tx_n,Tx_0)\\
&\leq s\alpha\max\left\{d(x_n,x_0),d(x_n,Tx_n),d(x_0,Tx_0),d(Tx_n,T^2x_n),\frac{d(T^2x_n,Tx_0)}{2s},\right.\\
&\qquad\left.\frac{(d(x_0,Tx_n)+d(x_n,Tx_0))}{2s}\right\}\\
&= s\alpha\max\left\{d(x_n,x_0),d(x_n,x_{n+1}),d(x_0,Tx_0),d(x_{n+1},x_{n+2}),\right.\\
&\qquad\left.\frac{d(x_{n+2},Tx_0)}{2s},\frac{(d(x_0,x_{n+1})+d(x_n,Tx_0))}{2s}\right\},
\end{aligned}$$

holds for all $n\in\mathbb{N}$.
Taking limit as $n\to\infty$ in the above inequality, we get

$$d(x_0,Tx_0)\leq s\alpha\max\{d(x_0,Tx_0),\frac{d(x_0,Tx_0)}{2s}\}\leq s\alpha d(x_0,Tx_0).$$

Hence,

$$d(x_0,Tx_0)\leq s\alpha d(x_0,Tx_0)<d(x_0,Tx_0),$$

which is a contradiction. Hence, $x_0=Tx_0$, that is, x_0 is the fixed point of T.

Now we prove the uniqueness of the fixed point. Suppose that T has distinct fixed points say x_0 and x_1. Since, $\lambda(\alpha)d(x_0,Tx_0) \leq d(x_0,Tx_0) = d(x_0,x_0) = 0 \leq d^*(x_0,x_1)$

$d(x_0,x_1) = d(Tx_0,Tx_1) \leq s\alpha U_T(x_0,x_1)$, where,

$$U_T(x_0,x_1) = \max\left\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Tx_1), d(Tx_0,T^2x_0), \frac{d(T^2x_0,Tx_1)}{2s}, \frac{(d(x_1,Tx_0)+d(x_0,Tx_1))}{2s}\right\}$$

Thus,

$$\begin{aligned}
d(Tx_0,Tx_1) &\leq s\alpha \max\left\{d(x_0,x_1), d(x_0,Tx_)), d(x_1,Tx_1), d(Tx_0,T^2x_0),\right. \\
&\quad \left.\frac{(d(T^2x_0,Tx_1))}{2s}, \frac{(d(x_1,Tx_0)+d(x_0,Tx_1))}{2s}\right\} \\
&\leq s\alpha \max\left\{d(x_0,x_1), \frac{(d(x_0,x_))}{2s}, \frac{(s[d(x_1,x_0)+d(x_0,x_1)])}{2s}\right\} \\
&= s\alpha \max\left\{d(x_0,x_1), \frac{d(x_0,x_1)}{2s}, \frac{(d(x_1,x_0)+d(x_0,x_1))}{2s}\right\} \\
&\quad \leq s\alpha d(x_0,x_1).
\end{aligned}$$

Hence, $d(x_0,x_1) \leq s\alpha d(x_0,x_1) < d(x_0,x_1)$ which is a contradiction. Hence, $x_0 = x_1$. Therefore, T has unique fixed point. □

By, taking

$$U_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{(d(y,Tx)+d(x,Ty))}{2s}\} \text{ for all } x,y \in X$$

in Theorem 4.1, we get the following corollary.

Corollary 4.2 *Let (X,d) be d-sequentially complete Δ-symmetric quasi b-metric space with $s \geq 1$ and a mapping $T : X \to X$. If there is $\alpha \in [0,\frac{1}{s})$ such that*

$$\lambda(\alpha)d(x,Tx) \leq d^*(x,y) \Rightarrow d^*(Tx,Ty) \leq s\alpha \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{(d(y,Tx)+d(x,Ty))}{2s}\}$$

for all $x, y \in X$.
Then T *has a unique fixed point.*

Also, by considering $U_T(x,y) = d(x,y)$ in Theorem 4.1, we get the following corollary.

Corollary 4.3 *Let* (X,d) *be d-sequentially complete* Δ*-symmetric quasi b-metric space with* $s \geq 1$ *and a mapping* $T : X \to X$*. If there is* $\alpha \in [0, \frac{1}{s})$ *such that* $\lambda(\alpha)d(x,Tx) \leq d^*(x,y) \Rightarrow d(Tx,Ty) \leq s\alpha d(x,y)$ *for all* $x, y \in X$.
Then T *has a unique fixed point.*

Now, we give an example in support of our main result.

Example 4.2 *Let* $X = \{x,y,z\}$*, we defined mapping* $d : X \times X \to \mathbb{R}_0{}^+$ *by*
$d(x,x) = d(y,y) = d(z,z) = 0;$
$d(x,y) = 0.7;$
$d(x,z) = 3;$
$d(y,z) = 1.3;$
$d(y,x) = 1.1;$
$d(z,x) = 1.4;$
$d(z,y) = 1.25;$
for all $x,y,z \in X$.
We observe that the triangle inequality for a quasi-metric space fails as show below.
$d(x,z) = 3 > 0.7 + 1.3 = d(x,y) + d(y,z).$
Hence, d is not quasi-metric space. Therefore,
$d(x,z) \leq s[d(x,y) + d(y,z)] = 3 \leq s[0.7 + 1.3]s \geq 3/2.$
And also, symmetric properties in b-metric space fails as show below.
$d(x,y) = 0.7 \neq 1.1 = d(y,x)$ *and hence d is not a b-metric space.*
Therefore, (X,d) *is* Δ*-symmetric quasi b-metric space for any* $\Delta \geq \frac{5}{2} = 2.5$ *and* $s \geq \frac{3}{2} = 1.5$.
Consider a mapping $T : X \to X$ *defined as*

$$T(k) = \begin{cases} y & \text{if } k \in \{y,z\}, \\ z & \text{if } k \in x,. \end{cases}$$

In this example, particularly let

$$\alpha = 0.4 \text{ and } s = 1.5. \tag{4.12}$$

So, $\lambda(\alpha)d(y,Ty) \leq \lambda(\alpha)d(y,y) = 0 \leq 1.1 = d^*(y,x)$.
Also, note that $d(Ty,Tx) = d^*(Ty,Tx) = d^*(y,z) = 1.3 > s\alpha d(y,x)$ *for any* $\alpha \in [0, \frac{1}{s})$ *and* $s \geq \frac{3}{2}$. *So,*

$$d(Ty,Tx) = d^*(Ty,Tx) = d^*(y,z) = 1.3 > 0.66 = s\alpha d(y,x).$$

Hence, T is neither d-contraction nor Suzuki type d-contraction in quasi b-metric space (X,d). Moreover, $\min\{d(x,Tx), d(y,Ty) = \min\{d(x,z), d(y,y)\} = 0 \leq 2d(x,y)$ *and* $d(Tx,Ty) = d(z,y) = 1.25 > 0.7 = s\alpha d(x,y)$ *for any* $\alpha \in [0, \frac{1}{s})$ *and* $s \geq \frac{3}{2}$.
So, $d(Tx,Ty) = d(z,y) = 1.25 > 0.42 = s\alpha d(x,y)$, *implies that T is not a 2-basic contraction in quasi b-metric space (X,d).*
Now, we show that T is generalized Suzuki type-contraction on (X,d) by considering the following cases:

Case a: $d(Ty,Tx) = d(y,z) = 1.3$, *and* $d(Ty,Tx) \leq s\alpha U_T(y,x)$, *where*

$$U_T(y,x) = \max\left\{ d(y,x), d(y,Ty), d(x,Tx), d(Ty,T^2y), \frac{d(T^2y,Tx)}{2s}, \frac{(d(x,Ty)+d(y,Tx))}{2s} \right\}.$$

Thus,

$$\begin{aligned}
d(Ty,Tx) &\leq s\alpha \max\left\{ d(y,x), d(y,Ty), d(x,Tx), d(Ty,T^2y), \frac{(d(T^2y,Tx))}{2s}, \frac{(d(x,Ty)+d(y,Tx))}{2s} \right\} \\
&\leq s\alpha \max\left\{ d(y,x), d(y,y), d(x,z), d(y,y), \frac{(s[d(y,x)+d(x,z)])}{2s}, \frac{(s[d(x,y)+d(y,z)])}{2s} \right\} \\
&= s\alpha \max\left\{ d(y,x), d(y,y), d(x,z), d(y,y), \frac{(d(y,x)+d(x,z))}{2}, \frac{(d(x,y)+d(y,z))}{2} \right\} \\
&= s\alpha \max\{1.1, 0, 3, 0, \frac{(1.1+3)}{2}, \frac{(0.7+1.3)}{2}\} = 3s\alpha.
\end{aligned}$$

Hence, $d(Ty,Tx) \leq s\alpha U_T(y,x)$. *It holds for all* $\alpha \in [0,\frac{1}{s})$ *and* $s \geq \frac{3}{2}$.
By (4.12)*, we have*

$$d(Ty,Tx) = d(y,z) = 1.3 \leq 1.8 = s\alpha U_T(y,x).$$

Case b: $d(Tx,Ty) = d(z,y) = 1.25$ *and* $d(Tx,Ty) \leq s\alpha U_T(x,y)$, *where*

$$U_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(Tx,T^2x), \frac{(d(T^2x,Ty))}{2s}, \frac{(d(y,Tx)+d(x,Ty))}{2s}\right\}.$$

Thus,

$$\begin{aligned}
d(Tx,Ty) &\leq s\alpha \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(Tx,T^2x), \frac{(d(T^2x,Ty))}{2s}, \frac{(d(y,Tx)+d(x,Ty))}{2s}\right\} \\
&= s\alpha \max\left\{d(x,y), d(x,z), d(y,y), d(z,z), \frac{d(z,y)}{2s}, \frac{(d(y,z)+d(x,y))}{2s}\right\} \\
&\leq s\alpha \max\left\{d(x,y), d(x,z), d(y,y), d(z,z), \frac{s[d(z,x)+d(x,y)]}{2s}, \frac{s[d(y,z)+d(x,y)]}{2s}\right\} \\
&= \{0.7, 3, 0, 0, \frac{(1.4+0.7)}{2}, \frac{(1.3+0.7)}{2}\} = 3s\alpha.
\end{aligned}$$

Hence, $d(Tx,Ty) \leq s\alpha U_T(x,y)$ *holds for all* $\alpha \in [0,\frac{1}{s})$ *and* $s \geq \frac{3}{2}$.
By (4.12)*, we have*
$d(Tx,Ty) = d(z,y) = 1.25 \leq 1.8 = s\alpha U_T(x,y)$.
Case c: $d(Tx,Tz) = d(z,y) = 1.25$ *and* $d(Tx,Tz) \leq s\alpha U_T(x,z)$, *where*

$$U_T(x,z) = \max\left\{d(x,z), d(x,Tx), d(z,Tz), d(Tx,T^2x), \frac{(d(T^2x,Tz))}{2s}, \frac{(d(z,Tx)+d(x,Tz))}{2s}\right\}.$$

Thus,

$$\begin{aligned} d(Tx,Tz) &\leq s\alpha \max\left\{ d(x,z), d(x,Tx), d(z,Tz), d(Tx,T^2x), \frac{(d(T^2x,Tz))}{2s}, \frac{(d(z,Tx)+d(x,Tz))}{2s} \right\} \\ &= s\alpha \max\left\{ d(x,z), d(x,z), d(z,y), d(z,z), \frac{d(z,y)}{2s}, \frac{(d(z,z)+d(x,y))}{2s} \right\} \\ &\leq s\alpha \max\left\{ d(x,z), d(x,z), d(z,y), d(z,z), \frac{([d(z,x)+(x,y)])}{2}, \frac{([d(x,z)+d(z,y)])}{2} \right\} \\ &= s\alpha \{3, 3, 1.25, 0, \frac{(1.4+0.7)}{2}, \frac{(3+1.25)}{2}\} = 3s\alpha. \end{aligned}$$

Hence, $d(Tx,Tz) \leq s\alpha U_T(x,z)$ *holds for all* $\alpha \in [0, \frac{1}{s})$ *and* $s \geq \frac{3}{2}$.
By (4.12)*, we have*
$d(Tx,Tz) = d(z,y) = 1.25 \leq 1.8 = s\alpha U_T(x,z)$.
Case d: $d(Tz,Tx) = d(y,z) = 1.3$ *and* $d(Tz,Tx) \leq s\alpha U_T(z,x)$*, where*

$$U_T(z,x) = \max\left\{ d(z,x), d(z,Tz), d(x,Tx), d(Tz,T^2z), \frac{(d(T^2z,Tx))}{2s}, \frac{(d(x,Tz)+d(z,Tx))}{2s} \right\}.$$

Thus,

$$\begin{aligned} d(Tz,Tx) &\leq s\alpha \max\left\{ d(z,x), d(z,Tz), d(x,Tx), d(Tz,T^2z), \frac{(d(T^2z,Tx))}{2s}, \frac{(d(x,Tz)+d(z,Tx))}{2s} \right\} \\ &= s\alpha \max\left\{ d(z,x), d(z,y), d(x,z), d(y,y), \frac{d(y,z)}{2s}, \frac{(d(x,y)+d(z,z))}{2s} \right\} \\ &\leq s\alpha \max\left\{ d(z,x), d(z,y), d(x,z), d(y,y), \frac{(s[d(y,x)+d(x,z)])}{2s}, \frac{(s[d(x,z)+d(z,y)])}{2s} \right\} \\ &= s\alpha \max\{1.4, 1.25, 3, 0, \frac{(1.1+3)}{2}, \frac{(3+1.25)}{2}\} = 3s\alpha. \end{aligned}$$

Hence, $d(Tz,Tx) \leq s\alpha U_T(z,x)$ *holds for all* $\alpha \in [0, \frac{1}{s})$ *and* $s \geq \frac{3}{2}$.
By (14) we have,
$d(Tz,Tx) = d(y,z) = 1.3 \leq 1.8 = s\alpha U_T(z,x)$.

Case e: $d(Ty,Tz) = d(y,y) = 0$. *So,*

$$d(Ty,Tz) \leq s\alpha U_T(y,z) \Rightarrow 0 \leq s\alpha U_T(y,z).$$

Case f: $d(Tz,Ty) = d(y,y) = 0$. *So,* $d(Tz,Ty) \leq s\alpha U_T(z,y) \Rightarrow 0 \leq s\alpha U_T(z,y)$ *holds for all* $\alpha \in [0, \frac{1}{s})$ *and* $s \geq \frac{3}{2}$.
By (4.12)*, we have*
$d(Ty,Tz) = d(y,y) = 0 \leq 1.8 = s\alpha U_T(y,z)$ *and* $d(Tz,Ty) = d(y,y) = 0 \leq 1.8 = s\alpha U_T(z,y)$ *respectively. Hence,* T *is generalized Suzuki type-contraction mapping on quasi b-metric space* (X,d)*, for all* $\alpha \in [0, \frac{1}{s})$ *and* $s \geq \frac{3}{2}$.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

Kikkawa and Suzuki (2008) generalized Banach contraction principle by introducing Suzuki type contractions of metric spaces. In addition, Romaguera (2022) discussed fixed point results for generalized Ciric's contractions of quasi-metric spaces. Romaguera's investigations revealed that unlike the Banach contraction principle and Kannan-type fixed point theorem, a basic Suzuki-type fixed point theorem cannot be transported as it is in the setup of quasi-metric spaces. Recently, Ali ,et al (2023) introduced the notions of Suzuki type-contraction and studied existence and uniqueness of fixed-point results in the setting of quasi- metric space and established fixed point theorems for this class of mapping. In this thesis work, we define generalized Suzuki type-contraction mappings and established fixed point theorem generalized Suzuki type-contraction mappings in the setting of quasi b-metric space and proved the existence and uniqueness of fixed point. Our results extend and generalize the work of Ali et al., (2023) fixed point of Suzuki type-contraction in quasi-metric space to fixed point results of generalized Suzuki type-contraction in quasi b-metric spaces. We have also supported the main results of this thesis work by applicable example.

5.2 Future scope

There are some published results related to the existence of fixed-point theorem of mappings defined on quasi-metric space. The researcher believes the search for the existence and uniqueness of fixed point result of self-mapping satisfying generalized Suzuki type-contraction mapping condition in quasi b-metric space is an active area of study. So, any interested researchers can use this opportunity and conduct their research work in this area.

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