APPROXIMATION OF A COMMON SOLUTION OF MONOTONE INCLUSION PROBLEMS AND FIXED POINT PROBLEM OF NON-LINER MAPPINGS IN HILBERT SPACES

A RESEARCH SUBMITTED TO THE DEPARTMENTOF MATHEMATICS INPARTIAL FULFILLMENT FOR THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, Nesru Mahammad Abafita with student ID number RM/0697/15-0,the undersigned declare that, this thesis paper entitled that " Approximation of a common solution of monotone inclusion problems and fixed point problem of non-liner mappings in Hilbert spaces" is my own original work and it has not been submitted to any institution and University elsewhere for the award of any academic degree or like, where other sources of information that have been used or quoted, they have been indicated and acknowledged.

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Acknowledgment

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Abstract

In this thesis we introduced an iterative algorithm for approximating a common solution of monotone inclusion problem and fixed point point problem of non-linear mappings in Hilbert Spaces and proved a strong convergence of a sequence generated by proposed algorithm to a common solution of monotone inclusion problem of the sum of two monotone mappings and fixed point problem of pseudo pseudocontractive mapping in Hilbert spaces provided that the mappings are uniformly continuous which are sequentially weakly continuous. Finally, we applied our main results to find a minimum point of a convex function in Hilbert spaces. Our results extended and generalized many results in the literature.

Acronym

Throughout this research, we denotes the following.

- H is real Hilbert space.
- C is nonempty close and convex subset of Hilbert space.
- $\|.\|$ is the norm space.
- $\langle ., . \rangle$ is the inner product space.

Contents

Chapter 1

Introduction

1.1 Background of the study

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $||.||$. Let $A : H \to 2^H$ be a nonlinear mapping. The domain, range, zero, and graph of A are respectively the sets $Dom(A) = \{x \in H : Ax \neq \emptyset\}$, $R(A) = \{Ax : x \in Dom(A)\}$, $Zero(A) = \{x \in H : 0 \in Ax\}$ and $Gph(A) = \{(x,y) \in H \times H : y \in Ax\}$. A mapping $A : H \to 2^H$ is called monotone if for any $x, y \in H$ and $u \in Ax, v \in Ay$ we have

$$\langle u - v, x - y \rangle \geq 0. \tag{1.1}$$

A monotone mapping $A : H \to 2^H$ is called maximally monotone if $Gph(A)$ is not properly contained in the graph of any other monotone operator. The resolvent of A is given by $J_{\lambda A} = (I + \lambda A)^{-1}$, where I is the identity mapping on H and $\lambda > 0$.

Let $A, B : H \to 2^H$ be maximally monotone mappings. Consider the problem of finding $z \in H$ such that

$$0 \in Az + Bz. \tag{1.2}$$

We denote the solution set of (1.2) by $(A + B)^{-1}(0)$. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. For solving problem (1.2), we remark that several authors have studied different iterative schemes (see, for example, Tseng (2000), Svaiter (2018) and the references therein).

In 1979, Passty introduced (1979) Forward-backward splitting method which defines a sequence $\{x_n\}$ by

$$x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n, n \geq 1, \tag{1.3}$$

where $\{r_n\}$ is a sequence of positive numbers, A and B are maximal monotone mappings with $Dom(A) \subset Dom(B)$, and A is single valued. This method is essentially generalizations of the classical gradient method for constrained convex optimization and monotone variational inequalities, and inherit restrictions similar to those methods such as A is single valued. In general, this method provides weak convergence even with the restriction that A is single-valued. It fails to provide weak convergence results to the zero of the sum of the general maximal monotone mappings.

In 1979, Lions and Mercier (1979), introduced Peaceman-Rachford splitting method whose iteration method is given by

$$\begin{aligned} y_n &= (2(I+\lambda B)^{-1} - I)x_n, \\ z_n &= (2(I+\lambda A)^{-1} - I)y_n, \\ x_{n+1} &= (1-\rho_n)x_n + \rho_n z_n, n \ \geq \ 1, \end{aligned} \tag{1.4}$$

where $\lambda > 0$ is a fixed scalar, and $\{\rho_n\} \subset (0,1]$ is a sequence of relaxation parameters. They proved weak convergence of this sequence to the solution of problem (1.2) under certain conditions.

With regard to a strong convergence, several authors have studied different iterative schemes (see for example, Takahashi (2010) and the references therein) for a zero of the sum of monotone mappings A and B.

In addition to monotone inclusion problems many problems that arises in several branches of applied mathematics such as game theory, variational analysis, optimization and differential equations can be reduced to finding solutions of an equation

$$Tx = x \tag{1.5}$$

(see, e.g., Zegeye (2007) and Zhang (2008) and the references therein). The solutions to this equation are called fixed points of the mapping T. It has been viewed that many of the most important nonlinear map arising in applied sciences areas can reduced to finding the fixed points of a certain mapping. In particular, fixed point techniques have been applied in diversified fields, such as science, economics, and engineering. Consequently, many authors concentrate on providing iterative algorithms for approximation of fixed points of mappings when they exists or assuming existence (see, e.g., Mann(1953), Berinde (2007), Browder (1968), Khan (2008) and Krasnoselskii (1955)).

The well known method for approximating a fixed point of contraction mapping is the Picard iterations. However, this iteration method may not always converge to a fixed point of T, when T is nonexpansive mapping.
So, for approximating fixed points of the classes of mappings are more general than the class of contraction mappings. Many iterative schemes, such as Mann iteration, Halpern Iteration, Ishikawa iteration, are introduced by different authors (see, e.g., Mann (1953), Halpern (1964), Ishikawa (1974)).

Many authors have also constructed an iterative algorithms called hybrid Mann and hybrid Ishikawa algorithms to obtained strong convergence of the sequence proposed by their method of converging a fixed point of Lipschitz pseudocontractive mappings (see, e.g., Liu et al. (2011), Marino et al. (2009)).

Zegeye and Wega (2020), introduced a new class of mapping which is more general than the class of pseudocotractive mappings called pseudo-pseudocontractive and established an iterative algorithm which converges strongly to a fixed point of pseudo-pseudocontractive mapping provided that the mapping is T is uniformly continuous which is sequentially weakly continuous.
We also remark that several authors have studied an iterative algorithms for approximating a common fixed point of a finite family of nonlinear mappings (see, e.g. Bauschke (1996), Yao et al. (2007), Zhou (2008), Zegeye and Wega (2020)).

Inspired and motivated by the above research works the purpose of this thesis is to introduce a new iterative algorithm for approximating a common solution of the sum of two monotone mappings and fixed point problem a pseudo-pseudo contractive mapping in Hilbert spaces. Moreover, we give an application to the convex minimization problem. Our results extend and generalize many results in the literature.

Now, we recall some definitions that the researcher will need in the following sequel.

Definition 1.1.1 *Let $T : C \longrightarrow H$ be mapping,*

i) T is called $L-$ Lipschitz mapping with Lipschtz constant $L > 0$ if

$$||Tx - Ty|| \leq L||x - y||$$

for all $x, y \in C$. If $0 \leq L < 1$, then T is called contraction. If $L = 1$, then T is called nonexpansive.

ii) T is called pseudocontractive mapping if for all $x, y \in C$ we have that

$$\langle\, x - y\ ,\ \ Tx - Ty\rangle \leq ||x - y||^2.$$

iii) T is called to be $\alpha-$strictly pseudocontractive mapping, if there exists a constant $\alpha > 0$ such that for all $x, y \in C$,

$$\langle\, x - y\ ,\ \ Tx - Ty\rangle \leq ||x - y||^2 - \alpha||(x - y) - (Tx - Ty)||^2.$$

iv) T is called $T : C \longrightarrow H$ is said to be pseudo-pseudocontractive mapping provided that for each $x, y \in C$, we have:

$$\langle\, x - Tx, y - x\rangle \geq 0 \ \ implies \ \ \langle\, y - Ty, y - x\rangle.$$

We remark that the class pseudo-pseudocontractive mappings are more general than the classes of mappings mentioned in (i)-(iii) above.

Definition 1.1.2 *The operator T is called sequentially weakly continuous if for each sequence x_n,we have x_n converges weakly to x implies Tx_n converges to Tx.*

1.2 Statements of the Problem

Many problems that arises in several branches of applied mathematics can be reduced to finding solutions of the generalized inclusion,

$$0 \in Ax + Bx, \tag{1.6}$$

where $B : C \to 2^H$, is maximal and monotone mappings $A : C \to H$ and H is real Hilbert space and C is closed and convex nonempty subsets of H. This inclusion problem is quite general as it includes variational inequality problems, equilibrium problems, complementary problems, minimization problems, fixed point problems as special cases as a result it has been studied by many researchers (see, Lions and Mercier (1979), Takahashi (2010), and the references therein).

In addition, Iterative methods for approximating fixed points of nonexpansive mappings have received vast investigations due to its extensive and wide applications in a variety of applied areas of image recovery, inverse problem, convex feasibility problem, partial differential equations and signal processing (see, Noor (2012), Yao (2007), and the references therein). It is known that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems (see, Scherzer (1995)). Consequently, many researchers have studied iterative methods which converges strongly a common fixed point of a finite family of pseudocontractive mappings in Hilbert spaces (see, Zegeye (2011), Daman and Zegeye (2012), Zegeye and Wega (2020)).

Daman and H. Zegeye (2012), established and proved strong convergence of Halpern-Ishikawa iterative method to a fixed point of Lipschitz pseudocontractive mapping without assuming that the interior point of the set of common fixed points of the mappings is nonempty in Hilbert spaces, either on C or on T.

Recently, Zegeye and Wega in (2020), introduced an iterative scheme for a fixed point of Lipschitz pseudocontractive mappings and proved a sequence generated by their proposed algorithm converges strongly to a fixed point of the mapping in Hilbert spaces.

Moreover, many authors have studied the common solution problems of monotone inclusion problems and fixed point problems in the settings of Hilbert spaces. In 2013, Hecai (2013), studied the common solution for two monotone operators and a quasinonexpansive mapping in the framework of Hilbert spaces. In 2016, Jingling Zhang and Nan Jiango (2016) investigated hybrid algorithm for a common zero point of the sum of two monotone mappings which is also a fixed point of a family of countable quasi-nonexpansive mappings. However, an iterative algorithm which converges to a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces is not yet studied.

Inspired and motivated by the research works of Hecai (2013), Jingling Zhang and Nan Jiango (2016) and Zegeye and Wega (2020), now in this thesis the researcher was establish a new iterative algorithm for approximating a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces.

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this thesis was to study iterative algorithms for approximating a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces.

1.3.2 Specific Objectives

The specific objectives of this thesis is to:

- investigate an iterative algorithm for approximating a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces
- prove the sequence generated by the proposed algorithm is bounded in Hilbert spaces.
- apply our main result to solve the minimization problems.

1.4 Significance of the Study

The outcome of this study have the following importance:

- It generalized the study of solutions of monotone inclusion of the sum of two monotone mappings in Hilbert spaces.
- It generalized the study fixed points of pseudo-pseudocontractive mappings in Hilbert spaces.
- It may provide some background information for other researchers who want to conduct a research on related topics.

1.5 Delimitation of the Study

This study was delimited to study iterative algorithms for approximating a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces.

Chapter 2

Review of Related Literatures

For solving problem (1.2), we remark that several authors have studied different iterative schemes (see, for example, Tseng (2000), Svaiter (2018) and the references therein).

In 1979, Passty (1979) introduced *Forward-backward* splitting method which defines a sequence $\{x_n\}$ by

$$x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n, n \geq 1, \tag{2.1}$$

where $\{r_n\}$ is a sequence of positive numbers, A and B are maximal monotone mappings with $Dom(A) \subset Dom(B)$, and A is single valued. This method is essentially generalizations of the classical gradient method for constrained convex optimization and monotone variational inequalities, and inherit restrictions similar to those methods such as A is single valued. In general, this method provides weak convergence even with the restriction that A is single-valued. It fails to provide weak convergence results to the zero of the sum of the general maximal monotone mappings.

In 1979, Lions and Mercier (1979) introduced Peaceman-Rachford splitting method whose iteration method is given by

$$\begin{aligned} y_n &= (2(I + \lambda B)^{-1} - I)x_n, \\ z_n &= (2(I + \lambda A)^{-1} - I)y_n, \\ x_{n+1} &= (1 - \rho_n)x_n + \rho_n z_n, n \geq 1, \end{aligned} \tag{2.2}$$

where $\lambda > 0$ is a fixed scalar, and $\{\rho_n\} \subset (0, 1]$ is a sequence of relaxation parameters. They proved weak convergence of this sequence to the solution of problem (1.2) under certain conditions.

Fixed point results give conditions under mappings of fixed point theory in which the desired iterative method converges to the solution. In the last one century the theory of fixed point has been reached as a powerful and important tool in the study of nonlinear problems. Banach (1922), introduced an iterative algorithm called Picard iteration for the class of contraction mappings and given by:

$$x_0 \in C, x_{n+1} = Tx_n, \quad n \geq 0. \tag{2.3}$$

The sequence generated by algorithm converges strongly to a unique fixed point of contraction mapping. However, this method in general failed to converge if T is not a contraction mapping. For instance, the mapping $T : [0,1] \to [0,1]$ defined $T(x) = 1 - x$ has a unique fixed point $\frac{1}{2} \in [0,1]$, it failed to converge. As a result many researchers introduced different types of algorithms for approximating fixed points mappings in Hilbert spaces (see, e.g., Mann (1953), Halpern (1964), Ishikawa (1974)). For approximating a fixed point of Mann introduced an iterative algorithm called Mann iteration Mann (1954), for approximating fixed points of nonexpansive mappings and it is given by

$$x_0 \in C, x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \text{ for } n \geq 0, \tag{2.4}$$

where $\{\alpha\}$ is a real sequence in the interval $(0,1)$ satisfying certain conditions. However, it is worth mentioning that the sequence generated by this scheme does not always converge strongly to a fixed point of nonexpansive mapping T. To obtain strong convergence of this method to a fixed point of T one has to impose compactness assumption on C (see, e.g., Chidume (1981), Kirk (1981)). Halpern (1964) introduced an iterative scheme called Halpern-iteration and it is given by

$$u,\ x_0 \in C,\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,\ n \leq 1. \tag{2.5}$$

He proved that the sequence generated by algorithm (2.5) converges to a fixed point of nonexpansive mapping T. Ishikawa (1974), construct an iterative scheme called Ishikawa-iteration for approximating fixed points of the class of pseudocontractive mappings which is more general than the class of nonexpansive mapping and the scheme is given by

$$\begin{aligned} x_0 \in C,\ y_n &= (1 - \beta_n)x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n \text{ for } n \geq 0. \end{aligned} \tag{2.6}$$

Where $\{\beta_n\}$ and $\{\alpha_n\}$ are real sequences in the interval $(0,1)$ satisfying certain conditions. He proved that the sequence generated by algorithm (2.6) converges strongly to a fixed point T provided that T is Lipschitz pseudocontractive mapping and C is a compact convex subset of a Hilbert space H.

Zhou (2008), established an iterative algorithm called hybrid Ishikawa algorithm and proved that the sequence generated by his method converges strongly to a fixed point of Lipschitz- pseudocontractive mapping without imposing the condition that C is compact.

Several authors have also established different schemes for approximating a common fixed point of a finite family of nonlinear mappings (see, e.g., Bauschke (1996), Yao et al. (2007), Zhou (2008), Zegeye et al. (2011) ,Takele and Reddy (2017)).

Zegeye et al. (2011), introduced Ishikawa iterative algorithm and proved that the sequence proposed by their method converges to strong convergence a common fixed point of finite family of Lipschitz pseudocontractive mappings in the setting of Hilbert spaces provided that interior of the set of fixed points of the mappings is nonempty.

Takele and Reddy (2017), also approximates a fixed point of a family of non-self and non-expansive mapping in Hilbert spaces and they also proved weak and strong convergence theorems.

2.1 Preliminaries

First we recall some known definitions, Lemmas and Theorems which are used in our subsequent analysis and recall some properties of the projection. Let C be a nonempty closed and convex of a Hilbert space H. for $x \in H$, the projection mapping $P_c : H \to C$ is defined by

$$||P_c x - x|| = inf_{y \in c} ||x - y||, \tag{2.7}$$

hence, P_c satisfies: $||P_c x - P_c y||^2 \leq \langle P_c x - P_c y, x - y \rangle$, for all $x, y \in H$.

Lemma 2.1.1 *For all $x, y \in H$, it is known that the following inequality hold.*

i) $||x + y|| \leq ||x|| + ||y||$

ii) $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$

iii) $||x - y||^2 = ||x||^2 - ||y||^2 - 2\langle x - y, y \rangle$

iv) $||x + y||^2 = ||x||^2 + 2\langle x + y, y \rangle$

Lemma 2.1.2 *(Albert,1996). Let C be complex subset of a real Hilbert space H and let $x \in H$, then*

$$x_0 = P_c x \text{ if and only if } \langle z - x_0, x - x_0 \rangle \leq 0, \text{ for every } z \in C.$$

Lemma 2.1.3 *(Xu, 2002). Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers that satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \gamma_n \text{ for } n \geq n_0 \text{ where } \{\alpha_n\} \subseteq (0,1) \text{ and } \{\gamma_n\} \subseteq \mathbb{R}, \text{ satisfies } lim_{x \to \infty} \alpha_n = 0,$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } lim \quad sup_{n \to \infty} \gamma_n \leq 0. \text{ Then } lim_{n \to \infty} a_n = 0.$$

Lemma 2.1.4 *(Mainge, 2008). Let $\{a_n\}$, be sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\alpha_{ni} < \alpha_{ni+1}$, for $i \in N$..Then there exists a non decreasing sequence $\{m_k\} \subset N$,, such that $m_k \to \infty$,and the following properties are satisfied for numbers $k \in N$:*

$$\alpha_{mk} \leq \alpha_{mk+1} \text{ in fact } m_k = max\{j \leq k : \alpha_j \leq \alpha_{j+1}\}.$$

Lemma 2.1.5 *(Zegeye and Shahzad). Let H be a real Hilbert space. Then for all* $x_i \in H$ *and*$\alpha_i \in [0,1]$ *for* $i = 1,2,3,...n$, *such that* $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n = 1$, *then the following holds*
$\|\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0\leq i,j\leq n} \alpha_i \alpha_j \|x_i - x_j\|^2$.

Lemma 2.1.6 *(He,2006). Let C, be a nonempty, closed and convex subset of H Let* $r(x)$*, be a real valued function on H and defined* $K := \{x \in C : r(x) \leq 0\}$*. If K, is nonempty and r is L-Lipshitz continuous with* $L > 0$*, then* $||P_K x - x|| \geq \frac{1}{L} \max\{r(x), 0\}$*, for* $x \in C$*.*

Chapter 3

Methodology

This chapter contains study site and period, study design, source of information and mathematical procedures.

3.1 Study Area and Period

The study will be conducted from September 2023 to June 2024 in Jimma University under the department of mathematics and conceptually the study will be focused on studying an approximation of a common solution of variational inequality and monotone inclusion problems of nonlinear mappings in Hilbert spaces.

3.2 Study Design

In order to achieve the objectives of this study was employ analytical methods of designing.

3.3 Sources of Information

The relevant sources of information for this study was published articles, books of different mathematics which related to our research topic.

3.4 Methodology

In this thesis, we were follow the standard mathematical procedures given below:

- Establishing an iterative algorithm and constructing theorems for approximating a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces.
- Proving strong convergence of the sequence proposed by the method to a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in the setting of Hilbert spaces.
- Applying our main result to convex solve minimization problems.

Chapter 4

Main Results

In this section we introduce an iterative algorithm for approximating a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in the setting of Hilbert spaces.

In this section, we shall make use of the following assumptions:
Assumption 1:

A1: Let C be a nonempty closed convex subset of a real Hilbert space, H.

A2: Let $A : C \to H$ be an $\alpha-$ inverse-strongly monotone mapping.

A3: Let B be a maximal monotone operator on H such that the domain of B is included in C.

A4: Let $T : H \to H$ be uniformly continuous pseudo-pseudocontractive mappings which is sequentially weakly continuous on bounded subset of H.

A5: Let $\Omega = F(T)\bigcap(A+B)^{-1}(0) \neq \emptyset$.

A6: Let $\iota \in (0,1), \mu > 0$ and $\delta \in [\underline{\delta}, \bar{\delta}] \subset (0, \frac{1}{\mu})$

A7: Let $\{\alpha_n\} \subset (0, \varepsilon)$ for some constant real number $\varepsilon > 0$ be a real sequence such that,

$$lim_{n\to\infty}\alpha_n = 0, \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Remark 4.1 *Note that if we have $d(x_n) = d'(x_n) = 0$ for some $n \in N$, then we get, $x_n \in \Omega = F(T) \cap F(T_2)$, since we have $d(x_n) = x_n - z_n = 0$ implies that, $x_n = (1-\delta)x_n + \delta T x_n$ which gives us, $\delta(Tx_n - x_n) = 0$, and hence $Tx_n = x_n$. For the rest of the study we consider only the case that this equality does not hold.*

Lemma 4.0.1 *Suppose that the assumption $A4 - A_5$ hold, and $\{x_n\}, \{y_n\}, \{z_n\}$, $\{u_n\}, \{v_n\}$ are sequences, generated by Algorithm 1.Then the search rules in step two are well defined.*

Algorithm 1: For arbitrary $q_0, u \in H$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} x_n = J_{r_n}(q_n - r_n A q_n) \\ z_n = (1-\delta)x_n + \delta T x_n \text{ and } d(x_n) = x_n - z_n, \end{cases} \tag{4.1}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n\to\infty} r_n > 0$ and $r_n \leq 2\alpha$.

Step 2. Compute

$$y_n = x_n - \Upsilon_n d(x_n) \tag{4.2}$$

where, $\Upsilon_n = \iota^{j_n}$ such that j_n is the smallest nonnegative integer j satisfying

$$\langle \iota^j (d(x_n)) + T(x_n - \iota^j d(x_n)) - T x_n, d(x_n)\rangle \leq \mu \|d(x_n)\|^2,$$

Step 3. Compute

$$\begin{cases} p_n = P_{C_n} x_n, \\ w_n = \theta_n q_n + \beta_n x_n + \eta_n p_n, \end{cases} \tag{4.3}$$

where $C_n = \{x \in H : \langle y_n - T y_n, x - y_n\rangle \leq 0\}$,
and $\{\theta_n\}, \{\beta_n\}, \{\eta_n\} \subset [\rho, 1)$ for $\rho > 0$ such that $\beta_n + \theta_n + \eta_n = 1$ for all $n \geq 0$.

Step 4. Compute

$$q_{n+1} = \alpha_n g(q_n) + (1-\alpha_n) w_n, \tag{4.4}$$

where $g : H \to H$ is a contraction mapping with constant coefficient α.

Step 5. Set $n := n+1$ and go to **Step 1**.

Proof: Since $\iota \in (0,1)$, T is uniformly continuous on H, We have

$$\langle \iota^j(d(x_n)) + T(x_n - \iota^j d(x_n)) - Tx_n, d(x_n)\rangle \to 0 \text{ as } j \to \infty.$$

Moreover, since $\|d(x_n)\| > 0$ there exist a non-negative integers j_n, satisfying the inequalities in Step 2. □

Lemma 4.0.2 *Suppose that the assumption $A_1 - A_3$ hold. If $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}$ are sequences generated by Algorithm 1 then, $\langle x_n - Tx_n, d(x_n)\rangle = \frac{1}{\delta}\|d(x_n)\|^2$ and*

$$\langle x_n - T_2x_n, d(x_n)\rangle = \frac{1}{\delta}\|d'(x_n)\|^2 \tag{4.5}$$

Proof: From equations (4.1), we have, $z_n = (1-\delta)x_n + \delta Tx_n$ which gives us, $z_n - x_n = \delta(Tx_n - x_n)$ and hence

$$\frac{z_n - x_n}{\delta} = Tx_n - x_n \tag{4.6}$$

Thus, from the fact that $d(x_n) = x_n - z_n$, we get

$$\begin{aligned} \langle x_n - Tx_n, d(x_n)\rangle &= \langle \frac{x_n - z_n}{\delta}, x_n - x_n\rangle \\ &= \frac{1}{\delta}\langle x_n - z_n, x_n - z_n\rangle \\ &= \frac{1}{\delta}\|x_n - z_n\|^2 \end{aligned} \tag{4.7}$$

□

Lemma 4.0.3 *Suppose the assumptions $A4 - A6$ holds. Let $p \in \Omega$, let $h_n(x_n) = \langle y_n - Ty_n, x_n - y_n\rangle$. Then, $h_n(p) \leq 0, \quad g_n(p) \leq 0, \quad h_n(x_n) \geq \Upsilon_n(\frac{1}{\delta} - \mu)\|d(x_n)\|^2$. In particular, if $d(x_n) \neq 0$, then $h(x_n) > 0$ and $g(x_n) > 0$.*

Proof: For the fact that $p \in \Omega$, we have

$$\langle p - Tp, y_n - p\rangle \geq 0. \tag{4.8}$$

This inequality and the fact that T is pseudo-pseudocontractive mapping, we obtain

$$h_n(p) = \langle y_n - Ty_n, y_n - p\rangle \geq 0,$$

which gives us,

$$h_n(p) = \langle y_n - Ty_n, p - y_n\rangle \leq .0$$

In addition, from Step 2, of Algorithm 1, we have,

$$\begin{aligned} h_n(x_n) &= \langle y_n - Ty_n, x_n - y_n\rangle \\ &= \langle y_n - Ty_n, x_n - (x_n - \Upsilon_n d(x_n)\rangle \\ &= \langle y_n - Ty_n, d(x_n)\rangle \end{aligned}$$

Furthermore, from the inequalities in Step 2, we have,

$$\langle x_n - y_n + Ty_n - Tx_n, d(x_n)\rangle \leq \mu\|d(x_n)\|^2,$$

which implies

$$\langle y_n - Ty_n, d(x_n)\rangle \geq \langle x_n - Ty_n, d(x_n)\rangle - \mu\|d(x_n)\|^2 \tag{4.9}$$

From Lemma 4.0.2 and inequality above, we obtain

$$\langle y_n - Ty_n, d(x_n)\rangle \geq (\frac{1}{\delta} - \mu)\|d(x_n)\|^2 \tag{4.10}$$

By combining (4.9) and (4.10), we obtain,

$$h_n(x_n) \geq \Upsilon_n(\frac{1}{\delta} - \mu)\|d(x_n)\|^2.$$

□

Theorem 4.0.4 *Suppose the assumptions $A1 - A7$ hold. Then, the sequence $\{x_n\}$, generated by the Algorithm 1 is bounded in Hilbert space, H.*

Proof: Let $P \in \Omega$. Then, $Tp = J_{r_n}(p - r_n Ap) = p$. For the fact that J_{r_n} nonexpansive and T is $\alpha-$ inverse-strongly monotone mapping, we have

$$\begin{aligned}
||x_n - p||^2 &= ||J_{r_n}(q_n - r_nAq_n) - p||^2 \\
&= ||J_{r_n}(q_n - r_nAq_n) - J_{r_n}(p - r_nAp)||^2 \\
&\leq ||(q_n - r_nAq_n) - (p - r_nAp)||^2 \\
&= ||(q_n - p) - r_n(A - Ap)||^2 \\
&= ||q_n - p||^2 - 2r_n\langle q_n - p, Aq_n - Ap\rangle + r_n^2||Aqn - Ap||^2 \\
&\leq ||q_n - p||^2 - r_n(2\alpha - r_n)r_n^2||Aqn - Ap||^2,
\end{aligned}$$

which implies

$$||x_n - p|| \leq ||q_n - p||. \tag{4.11}$$

In addition from Lemma 2.1.5, equation (4.11) and equation (2.7), we obtain

$$\begin{aligned}
\|q_{n+1} - p\| &= \|\alpha_n g(q_n) + (1 - \alpha_n)w_n - p\| \\
&= \|\alpha_n g(q_n) + (1 - \alpha_n)w_n - [\alpha_n p + (1 - \alpha_n p)]\| \\
&\leq \alpha_n\|g(q_n) - g(p)\| + \alpha_n\|g(p) - p\| + \|(1 - \alpha_n)(w_n - p)\| \\
&= \alpha_n\alpha\|q_n - p\| + \alpha_n\|g(p) - p\| + (1 - \alpha_n)\|\theta_n q_n + \beta_n x_n + \eta_n p_n - p\| \\
&= \alpha_n\alpha\|q_n - p\| + \alpha_n\|g(p) - p\| \\
&\quad + (1 - \alpha_n)[\theta_n\|(q_n - p)\| + \beta_n\|(x_n - p)\| + \eta_n\|p_n - p\|] \\
&= \alpha_n\alpha\|q_n - p\| + \alpha_n\|g(p) - p\| \\
&\quad + (1 - \alpha_n)[\theta_n\|q_n - p\| + \beta_n\|x_n - p\| + \eta_n\|P_{C_n}x_n - p\|] \\
&\leq \alpha_n\alpha\|q_n - p\| + \alpha_n\|g(p) - p\| \\
&\quad + (1 - \alpha_n)[\theta_n\|q_n - p\| + \beta_n\|x_n - p\| + \eta_n\|x_n - p\|] \\
&\leq \alpha_n\alpha\|q_n - p\| + \alpha_n\|g(p) - p\| + (1 - \alpha_n)\|q_n - p\|] \\
&= (1 - \alpha_n(1 - \alpha))\|q_n - p\| + \alpha_n\|g(p) - p\| \\
&\leq \max\left\{\|q_n - p\|, \frac{g(p) - p}{1 - \alpha}\right\}
\end{aligned}$$

Hence, by induction

$$\|q_{n+1} - p\| \leq \max\left\{\|q_0 - p\|, \frac{g(p) - p}{1 - \alpha}\right\}.$$

Thus, the sequence $\{q_n\}$ is bounded and hence the sequences $\{z_n\}, \{y_n\}, \{Tx_n\}$ are bounded. □

Theorem 4.0.5 *Suppose the assumption $A1-A7$ hold. Then the sequence $\{q_n\}$, generated by the algorithm 1, converges strongly to $p = P_\Omega g(p)$.*

Proof: Now, let $p = P_\Omega g(p)$. From equation 2.7, we have,

$$\|p - p_n\|^2 \leq \|p - x_n\|^2 - \|x_n - p_n\|^2.$$

Since T is bounded on bounded subset of H ,Then their exists $L > 0$, such that

$$\|Ty_n - y_n\| \leq L,$$

for all $n \geq 0$. Thus,

$$\begin{aligned}
|h_n(z) - h_n(w)| &= |\langle y_n - Ty_n, z - y_n\rangle - \langle y_n - Ty_n, z - y_n\rangle| \\
&= |\langle y_n - Ty_n, z - w\rangle \\
&\leq |y_n - Ty_n\| \|z - w\| \\
&\leq L\|z - w\|,
\end{aligned}$$

which gives us that h_n is L- Lipschitz continuous on H. Thus, from Lemma 2.1.6 and Lemma 4.0.3, we obtain

$$\|x_n - p_n\|^2 \geq \frac{h_n x_n}{2L^2} \geq \Upsilon_n^2(\frac{1}{\delta} - \mu)^2 \|d(x_n)\|^4 \tag{4.12}$$

Thus, from (4.12), we get

$$\|p - p_n\|^2 \leq \|p - x_n\|^2 - \Upsilon_n^2(\frac{1}{\delta} - \mu)^2 \|d(x_n)\|^4 \tag{4.13}$$

Now, rom Lemma 2.1.5, from ((4.13)) and (4.11), we get

$$\begin{aligned}
\|w_n-p\|^2 &= \|\theta_n q_n+\beta_n x_n+\eta_n p_n-p\|^2\\
&= \|\theta_n(q_n-p)+\beta_n(x_n-p)+\beta_n(p_n-p)\|^2\\
&\le \theta_n\|q_n-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|p_n-p\|^2\\
&\quad -\theta_n\beta_n\|q_n-x_n\|^2-\theta_n\eta_n\|q_n-p_n\|^2-\eta_n\beta_n\|p_n-x_n\|^2\\
&\le \theta_n\|q_n-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|x_n-p\|^2\\
&\quad -\Big(\Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d(x_n)\|^4\Big)\\
&\quad -\Big(\theta_n\beta_n\|p_n-p_n\|^2+\theta_n\eta_n\|x_n-q_n\|^2+\theta_n\beta_n\|p_n-q_n\|^2\Big)\\
&\le \|p-q_n\|^2-\Big(\Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d(x_n)\|^4\Big)\\
&\quad -\Big(\theta_n\beta_n\|q_n-x_n\|^2+\theta_n\eta_n\|q_n-p_n\|^2+\eta_n\beta_n\|p_n-x_n\|^2\Big)
\end{aligned} \tag{4.14}$$

By Lemma 2.1.1, Lemma 2.1.5 and the above inequality, we obtain

$$\begin{aligned}
\|q_{n+1}-p\|^2 &= \|\alpha_n g(q_n)+(1-\alpha_n)w(n)-p\|^2\\
&\le \|\alpha_n(g(q_n)-g(p))+(1-\alpha_n)(w_n-p)\|^2+2\alpha_n\langle g(p)-p,q_{n+1}-p\rangle\\
&\le \big(1-(1-\alpha)\alpha_n\big)^2\|q_n-p\|^2+2\alpha_n\|g(q_n)-p\|\|q_{n+1}-q_n\|\\
&\quad +2\alpha_n\langle g(p)-p,q_n-p\rangle\\
&\le \big(1-(1-\alpha)\alpha_n\big)\|p-q_n\|^2+2\alpha_n\|g(q_n)-p\|\|q_{n+1}-a_n\|\\
&\quad +2\alpha_n\langle g(p)-p,q_n-p\rangle\\
&\quad -\big(1-(1-\alpha)\alpha_n\big)\Big(\Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d(q_n)\|^4\Big)\\
&\quad -\big(1-(1-\alpha)\alpha_n\big)\Big(\theta_n\beta_n\|q_n-x_n\|^2+\theta_n\eta_n\|q_n-p_n\|^2+\eta_n\beta_n\|p_n-x_n\|^2\Big),
\end{aligned}$$

which gives us

$$\begin{aligned}
&\big(1-(1-\alpha)\alpha_n\big)\Big(\theta_n\beta_n\|q_n-x_n\|^2+\theta_n\eta_n\|q_n-p_n\|^2+\eta_n\beta_n\|p_n-x_n\|^2\Big)\\
&+(\frac{1}{\delta}-\mu)^2[\Upsilon_n^2\beta_n\|d(x_n)\|^4\\
\le\ & \|q_n-p\|^2-\|q_{n+1}-p\|^2+2\alpha_n\|g(q_n)-p\|\|q_{n+1}-q_n\|\\
&+2\alpha_n\langle g(p)-p,q_n-p\rangle.
\end{aligned} \tag{4.15}$$

Next, we that the sequence $\{\|p_n - p\|\}$ converges strongly to zero. For this we consider two cases as follows:

Case 1: Assume that there exist $n_0 \in N$, such that the sequence of real numbers $\{\|p_n - p\|^2\}$ is decreasing for all $n \geq n_0$. Thus, the sequence $\{\|p_n - p\|^2\}$ convergent and hence from (4.15) and the fact that $\alpha_n \to 0$, we obtain

$$\lim_{n\to\infty} \|q_n - x_n\|^2 = \lim_{n\to\infty} \|q_n - p_n\|^2 = \lim_{n\to\infty} \|p_n - x_n\|^2 = 0.$$

In addition, we have

$$\lim_{n\to\infty} \Upsilon_n^2 \|d(x_n)\|^4 = 0.$$

Then, from this we obtain that

$$\lim_{n\to\infty} \Upsilon_n \|d(x_n)\|^2 = 0. \tag{4.16}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$, of $\{x_n\}$ which converges weakly to $q \in H$ and

$$\limsup_{n\to\infty} \langle g(p) - p, x_n - p\rangle = \lim_{k\to\infty} \langle g(p) - p, x_{n_k} - p\rangle. \tag{4.17}$$

Now, we prove that

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0, \lim_{k\to\infty} \|x_{n_k} - u_{n_k}\| = 0 \tag{4.18}$$

First consider the case, when $\liminf_{k\to\infty} \Upsilon_{n_k} > 0$ In this case there is $\Upsilon > 0$ such that $\Upsilon_{n_k} > \Upsilon > 0$, for all $k \in N$. Thus, we have

$$\|x_{n_k} - z_{n_k}\|^2 = \frac{1}{\Upsilon_{n_k}} \Upsilon_{n_k} \|x_{n_k} - z_{n_k}\|^2 \leq \frac{1}{\Upsilon} \Upsilon_{n_k} \|x_{n_k} - z_{n_k}\|^2.$$

From this inequality and ((4.16)), we obtain

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\|^2 = 0$$

and hence

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0$$

. **S**econd consider, when $\lim inf_{k\to\infty}\Upsilon_{n_k}=0$. In this case

$$\lim_{k\to\infty}\Upsilon_{n_k}=0 \text{ and } \lim_{k\to\infty}\|x_{n_k}-z_{n_k}\|^2=c>0 \tag{4.19}$$

Consider, $y'_{n_k}=\frac{1}{\iota}\Upsilon_{n_k}z_{n_k}+(1-\frac{1}{\iota}\Upsilon_{n_k})x_{n_k}$
Thus, from (4.19), we have

$$\lim_{k\to\infty}\|y'_{n_k}-z_{n_k}\|=\lim_{k\to\infty}\frac{1}{\iota}\Upsilon_{n_k}\|.x_{n_k}-z_{n_k}\|=0 \tag{4.20}$$

From inequality in Step 2 and definition of y'_{n_k}, we obtain

$$\begin{aligned}\mu\|x_{n_k}-z_{n_k}\|^2 &< \langle x_{n_k}-y'_{n_k}+Ty'_{n_k}-Tx_{n_k},x_{n_k}-z_{n_k}\rangle\\ &\leq \langle x_{n_k}-y'_{n_k},x_{n_k}-z_{n_k}\rangle+\langle Ty'_{n_k}-Tx_{n_k},x_{n_k}-z_{n_k}\rangle\\ &\leq \|x_{n_k}-y'_{n_k}\|\|x_{n_k}-z_{n_k}\|+\|+Ty'_{n_k}-Tx_{n_k}\|\|x_{n_k}-z_{n_k}\|\end{aligned} \tag{4.21}$$

From (4.20), (4.21) and the fact that T is uniformly continuous, we get $\lim_{n\to\infty}\|x_{n_k}-z_{n_k}\|=0$, which contradict (4.19). Thus, from this fact the equation (4.18) holds. Furthermore, from Step 1 of the Algorithm 1, we have

$$z_{n_k}=(1-\delta)x_{n_k}+\delta Tx_{n_k},$$

which gives as

$$\|z_{n_k}-x_{n_k}\|=\delta\|x_{n_k}-Tx_{n_k}\|. \tag{4.22}$$

Hence, from equation (4.18), we obtain,

$$\lim_{k\to\infty}\|x_{n_k}-Tx_{n_k}\|=0. \tag{4.23}$$

From (4.23) and the fact that T is sequentially weakly continuous, we get $q\in F(T)$. Moreover, since

$$x_{n_k}=J_{r_{n_k}}(q_{n_k}-r_{n_k}Aq_{n_k}),$$

we get,

$$q_{n_k}-r_{n_k}Aq_{n_k}\in x_{n_k}+Bx_{n_k}, \tag{4.24}$$

which implies

$$\frac{q_{n_k} - x_{n_k}}{r_{n_k}} - Aq_{n_k} \in Bx_{n_k}. \tag{4.25}$$

For B is maximal monotone, so we get for any $(s,t) \in B$,

$$\langle x_{n_k} - s, \frac{q_{n_k} - x_{n_k}}{r_{n_k}} - Aq_{n_k} - t \rangle \geq 0. \tag{4.26}$$

By letting $k \to \infty$, we get

$$\langle q - s, -Aq - t \rangle \geq 0. \tag{4.27}$$

Since B is a maximal monotone operator, so we have $-Aq \in Bq$ and hence $0 \in (A + B)^{-1}(q)$. Therefore, $q \in \Omega$. From the definition of q_{n+1}, we have $\|q_{n+1} - w_n\| = \alpha_n \|u - w_n\| \to 0$, as $n \to \infty$, since $\alpha_n \to \infty$, as $n \to \infty$ and from (4.15), we get

$$\begin{aligned} \|q_{n+1} - w_n\| &\leq \|q_{n+1} - w_n\| + \|w_n - q_n\| \\ &\leq \|q_{n+1} - w_n\| + \|\theta_n q_n + \beta_n x_n + \eta_n p_n - q_n\| \\ &\leq \|q_{n+1} - w_n\| + \theta_n \|q_n - q_n\| & (4.28) \\ &\quad + \beta_n \|x_n - q_n\| + \eta_n \|p_n - q_n\| \to 0 \text{ as } n \to 0. & (4.29) \end{aligned}$$

Moreover,

$$\|x_n - w_n\| \leq \theta_n \|q_n - q_n\| + \beta_n \|x_n - q_n\| + \eta_n \|p_n - q_n\| \to 0 \text{ as } n \to 0. \tag{4.30}$$

Thus, from (4.28) and (4.30), we obtain

$$\begin{aligned} \|q_{n+1} - q_n\| &= \|q_{n+1} - w_n + w_n - q_n\| \\ &\leq \|q_{n+1} - w_n\| + \|w_n - q_n\| \to 0, \text{ as } n \to 0. & (4.31) \end{aligned}$$

From (4.17) and Lemma 2.1.2, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle g(p) - p, q_n - p \rangle &\leq \lim_{k\to\infty} \langle u - q, q_{n_k} - p \rangle \\ &= \langle g(p) - p, q - p \rangle \\ &\leq 0. & (4.32) \end{aligned}$$

Now, we show that the sequence $\{x_n\}$ converges strongly to p. Indeed, from Lemma

2.1.1 and 4.14, we obtain

$$\begin{aligned}
\|q_{n+1}-p\|^2 &= \|\alpha_n g(q_n)+(1-\alpha_n)w_n-[\alpha_n p+(1-\alpha_n)p\|^2 \\
&\leq (1-\alpha_n)^2\|w_n-p\|^2+2\alpha_n\langle g(q_n)-p,q_{n+1}-p\rangle \\
&\leq (1-\alpha_n)^2\|w_n-p\|^2+2\alpha_n||g(q_n)-p||||q_{n+1}-p|| \\
&\leq (1-\alpha_n)^2\|q_n-p\|^2+2\alpha\alpha_n||q_n-p||||q_{n+1}-p|| \\
&\quad +2\alpha_n\langle g(p)-p,q_{n+1}-p\rangle \\
&\leq (1-2\alpha_n+2\alpha\alpha_n)\|q_n-p\|^2+\alpha_n^2\|q_n-p\|^2 \\
&\quad +2\alpha_n\langle g(p)-p,q_{n+1}-p\rangle \\
&= (1-(1-\alpha)2\alpha_n)\|q_n-p\|^2+\alpha_n^2\|q_n-p\|^2 \\
&\quad +2\alpha_n\langle g(p)-p,q_{n+1}-p\rangle \\
&= (1-2\alpha_n(1-\alpha))\|q_n-p\|^2 \\
&\quad +2\alpha_n(1-\alpha)\Big[\frac{\alpha_n\|q_n-p\|^2}{2(1-\alpha)}+\frac{\langle g(p)-p,q_{n+1}-p\rangle}{1-\alpha}\Big]
\end{aligned} \tag{4.33}$$

Finally, from (4.33), (4.31), (4.32) and Lemma 2.1.3, we get $\|q_n-p\|^2\to 0$, as $n\to\infty$ and hence $q_n\to p$, as $n\to\infty$.

Case 2: Suppose that there exists a subsequence $\{\|q_{n_i}-p\|^2\}$ of $\{\|q_n-p\|^2\}$ such that

$$\|q_{n_i}-p\|^2<\|q_{n_i+1}-p\|^2,\ \text{for } i\geq 0. \tag{4.34}$$

Thus by Lemma 2.1.4 there exists a non-decreasing sequence $\{m_k\}$,of the set of positive integer of numbers such that $m_k\to 0$, as $k\to\infty$,
$\|q_{m_k}-p\|^2\leq\|q_{m_k+1}-p\|^2$ and

$$\|q_k-p\|^2\leq\|q_{m_k+1}-p\|^2,\ \text{for all } k\geq 1. \tag{4.35}$$

Hence, by following the method of Case 1, from the inequality (4.17), we obtain

$$\lim_{k\to\infty}\|q_{m_k}-p_{m_q}\|=\lim_{k\to\infty}\|x_{m_k}-q_{m_k}\|=0.$$

In addition,

$$\lim_{k\to\infty}\|q_{m_k}-z_{m_k}\|=0=\lim_{k\to\infty}\|q_{m_k+1}-q_{m_k}\|=0,$$

and

$$\limsup_{k\to\infty}\langle u-p, q_{m_k+1}-p\rangle \leq 0. \tag{4.36}$$

Now, from (4.33), we get

$$\begin{aligned}
\|q_{m_k+1}-p\|^2 &\leq (1-2\alpha_{m_k}(1-\alpha))\|q_{m_k}-p\|^2 \\
&\quad +2\alpha_{m_k}(1-\alpha)\Big[\frac{\alpha_{m_k}\|q_{m_k+1}-p\|^2}{2(1-\alpha)}+\frac{\langle g(p)-p, q_{m_k+1}-p\rangle}{1-\alpha}\Big] \qquad (4.37)\\
&\leq (1-2\alpha_{m_k}(1-\alpha))\|q_{m_k+1}-p\|^2 \\
&\quad +2\alpha_{m_k}(1-\alpha)\Big[\frac{\alpha_{m_k}\|q_{m_k+1}-p\|^2}{2(1-\alpha)}+\frac{\langle g(p)-p, q_{m_k+1}-p\rangle}{1-\alpha}\Big], \qquad (4.38)
\end{aligned}$$

which implies that

$$\begin{aligned}
\|q_k-p\|^2 &\leq \|q_{m_k+1}-p\|^2 \\
&\leq \Big[\frac{\alpha_{m_k}\|q_{m_k+1}-p\|^2}{2(1-\alpha)}+\frac{\langle g(p)-p, q_{m_k+1}-p\rangle}{1-\alpha}\Big].
\end{aligned} \tag{4.39}$$

Therefore, from (4.36) and (4.39), we conclude that $\limsup_{k\to}\|q_k-p\| \leq 0$, that is, $q_k \to p$ as, $k \to \infty$. □

We remark that from Theorem 4.0.5 we obtain the following the following result for two pseudocontractive mappings which are sequentially weakly continuous on bounded subset of H.

Corollary 4.0.6 *Suppose the assumption $A1-A3, A5, A6$ and $A7$ hold. Let $T: H \to H$ be uniformly continuous pseudocontractive mappings which are sequentially weakly continuous on bounded subset of H. Then the sequence $\{q_n\}$ generated by the Algorithm 1, converges strongly to $p = P_\Omega g(p)$.*

We remark that from Theorem **??** we obtain the following the following result for a finite family of pseudocontractive mappings which are sequentially weakly continuous on bounded subset of H.

4.1 Application

In this section, we apply Corollary 4.40 to find a common minimum point of a finite family of convex functions in Hilbert spaces.

Let $f : H \to \mathbb{R}$ be a convex smooth function. We consider the problem of finding a point of $z \in H$ such that

$$f(z) = \min_{x \in H}\{f(x)\}. \tag{4.40}$$

By Fermat's rule this problem is equivalent to the problem of finding $z \in H$ such that

$$\nabla fz, \tag{4.41}$$

where ∇f is a gradient of f. We note that ∇f is monotone where ∇f is a gradient of f (see, Rockafellar (1970).

We note that the mapping T is pseudocontractive T if and only if $A := I - T$, where I is the identity mapping on H is monotone mapping. Thus, one way of solving problem 4.40 is finding the fixed point of pseudocontractive T for the fact that the set of fixed points of T, $F(T) = \{x \in H : Tx = x\}$, is the set of zero points A, $N(A) = \{x \in H : Ax = 0\}$.

Now, if in Algorithm 2, we assume $\nabla f = I - T$, then we obtain the following Algorithm **??** for a minimum point problem of a convex function in Hilbert spaces.

The method of proof Theorem 4.0.4 provides the proof of the following theorem of finding a common solution of monotone inclusion problem and minimum point problem of a convex function in Hilbert spaces.

Theorem 4.1.1 *Suppose the Assumptions (A1, A2, A3, A6) and (A7) hold. Let $f : E \to \mathbb{R}$ be a convex smooth functions with ∇f are uniformly continuous which are sequentially weakly continuous on bounded subset of H and $\Omega = \mathbb{F} \bigcap (A + B)^{-1}(0) \neq \emptyset$, where $\mathbb{F} = \{z : f(z) = \min_{x \in H} f(x)\}$. Then, the sequens $\{q_n\}$ generated by Algorithm 2 converges strongly to an element $p = P_{\Omega}(u)$.*

Algorithm 2: For arbitrary $q_0, u \in H$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} x_n = J_{r_n}(q_n - r_n A q_n) \\ z_n = (1-\delta)x_n + \delta(x_n - Tx_n) \text{ and } d(x_n) = x_n - z_n, \end{cases} \tag{4.42}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n\to\infty} r_n > 0$ and $r_n \leq 2\alpha$.

Step 2. Compute

$$y_n = x_n - \Upsilon_n d(x_n) \tag{4.43}$$

where, $\Upsilon_n = \iota^{j_n}$ such that j_n is the smallest nonnegative integer j satisfying

$$\langle \iota^j(d(x_n)) + T(x_n - \iota^j d(x_n)) - (x_n - Tx_n), d(x_n)\rangle \leq \mu \|d(x_n)\|^2,$$

Step 3. Compute

$$\begin{cases} p_n = P_{C_n} x_n, \\ w_n = \theta_n q_n + \beta_n x_n + \eta_n p_n, \end{cases} \tag{4.44}$$

where $C_n = \{x \in H : \langle y_n - (y_n - Ty_n), x - y_n\rangle \leq 0\}$,
and $\{\theta_n\}, \{\beta_n\}, \{\eta_n\} \subset [\rho, 1)$ for $\rho > 0$ such that $\beta_n + \theta_n + \eta_n = 1$ for all $n \geq 0$.

Step 4. Compute

$$q_{n+1} = \alpha_n g(q_n) + (1-\alpha_n) w_n, \tag{4.45}$$

where $g : H \to H$ is a contraction mapping with constant coefficient α.

Step 5. Set $n := n+1$ and go to **Step 1**.

Proof: Put $T = \nabla f$. Thus, we have that the mappings T is uniformly continuous which are sequentially weakly continuous on bounded subset of H and hence Algorithm 2 reduces to Algorithm 1. Thus, Theorem 4.0.5 provides the conclusion of the theorem. □

Chapter 5

Conclusion and Future scope

5.1 Conclusion

In this thesis, we established an iterative algorithm for approximating a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces.
In addition, we also proved a strong convergence of a sequence generated by the proposed algorithm to a a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces. Our result generalizes many results in the literature.

5.2 Future Scope

In this thesis we obtained results on approximating a common solution of monotone inclusion of the sum of two monotone mappings and fixed point problem of a pseudo-pseudocontractive mapping in Hilbert spaces. However, extending this result to a Banach spaces more general Hilbert spaces is an open problem. So, any interested researchers can use this opportunity to conduct their research work in this area.

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